Universidade Técnica de Lisboa

Instituto Superior Técnico

Departamento de Matemática

Algebraization of logics and beyond

Ricardo João Rodrigues Gonçalves

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Supervisor:
Prof. Carlos Caleiro

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Chapter 1

Introduction

Algebraic logic in modern sense was born with the work of Tarski, in particular with his 1935 paper (see [Tar83]) where we can find for the first time some characteristic features of the subject we recognize today. In this paper, he gave the precise connection between Boolean algebra and classical propositional logic, using the idea of looking at the set of formulas as an algebra with operators induced by the logical connectives. He observed that logical equivalence was a congruence on the formula algebra, and a quotient algebra could be built. This is the so-called Lindenbaum-Tarski method. It turns out that the quotient algebra is a Boolean algebra, and the theorems coincide exactly with the formulas equivalent to $\top$.

Using this idea, a number of other logics were algebraized, namely the intuitionistic propositional logic of Heyting, the multiple-valued logics of Post and Lukasiewicz and the modal logics S_4 and S_5 of Lewis. In contrast to Boolean, cylindric, polyadic and Wajsberg algebras which were known before the Lindenbaum-Tarski method was first applied to generate them from the appropriate logics, Heyting algebras were first identified precisely by applying the Lindenbaum-Tarski method to intuitionistic propositional logic.

Although the focus of algebraic logic was on finding an algebraic counterpart for particular classes of logics, there was also interest, when this counterpart was found, in investigating the relationship between the metalogical properties of the logic and the algebraic properties of the algebraic counterpart. These results are usually called bridge theorems and allow us to use powerful methods of modern algebra in the investigation of metalogical properties of algebraizable logics.

The investigation of particular classes of logics gave place to a systematic investigation of broad classes of logics in a more abstract context. The focus has turned to the process of algebraization itself rather than being centered on the algebraization of particular classes of logics. Only in the 1989 monograph by Blok and Pigozzi [BP89], the concept of algebraizable logic was given a mathematical precise sense.

The classical approach considers that the formulas are elements of a free algebra and the interpretation structures are logical matrices [Wój88] based on algebras of the same type and where the value of each formula is calculated homomorphicaly. But this approach has inherently some limitations of applicability, namely to non-truth-functional logics, where there are constructors for which the principle of substitution of equivalents is not valid. The paradigmatic examples of non-truth-functional logics are the paraconsistent logics of da Costa [dC63, dC74].

The main motivation of this work is precisely to extend the existing theory of algebraization in order to encompass also these non-truth-functional logics.

Our objective is first to give an alternative characterization of the main concepts of the theory introduced by Blok and Pigozzi, using maps between the target logic and unsorted equational logic. With this characterization, we can then replace unsorted equational logic by another base logic. We then characterize the properties that this base logic has to satisfy in order to maintain some "equational flavor", that is, we try to abstract away the properties of unsorted equational logic that are essential to the process of algebraization. Another objective is the study of the example of paraconsistent logic $\mathcal{C}_1$ of da Costa using this new approach. $\mathcal{C}_1$ was shown to be non-algebraizable first in [Mor80] and recently using the Blok and Pigozzi theory in [LMS91]. For this particular example, we replace unsorted equational logic by two-sorted equational logic, with sorts of formulas and truth-values, using ideas in [CCC$^+$03], and prove that $\mathcal{C}_1$ is algebraizable in this new context. We refer to [BS81] with respect to details on universal algebra. In Chapter 2 we introduce some basic concepts. We define our working universe of logics and define the notion of a map of logics, an essential ingredient to the generalization we aim to achieve. We also prove some useful lemmas about maps of logics and, in the last section, we recall some examples of logics. Then, in Chapter 3, we recall the main concepts of the theory of algebraization of logics introduced by Blok and Pigozzi [BP89]. We then give an alternative characterization of some of the notions previously introduced by means of the existence of maps between the logic in study and unsorted equational logic. This is the (crucial) first step toward a generalization of these concepts, that will be done in Chapter 4. This generalization is based on substituting unsorted equational logic by another logic. In this chapter we try to characterize the properties that this replacing logic has to satisfy in order to guarantee that the process of algebraization still works. We also try to establish some natural analogue results to those introduced in Chapter 3. Finally, in Chapter 5, we study in detail the application of this new theory to the concrete example of $\mathcal{C}_1$ of DaCosta.

Chapter 2

Logics

In this chapter we will introduce some important definitions, that will be essential for the consequent chapters. We begin by the general definition of a *logic* and define what we mean by *map between logics* giving some useful lemmas. Then we present a particular case of logics, the *structural propositional logics*. In the last section of this chapter we will give some examples of logics.

2.1 Logical consequence

Definition 2.1.1 (Logic)
A *logic* is a pair $\mathcal{L} = \langle L, \vdash \rangle$, where L is a set (of *formulas*) and $\vdash \subseteq 2^L \times L$ is a *consequence relation* satisfying the following conditions [Tar83]:

Reflexivity: if $\varphi \in \Gamma$ then $\Gamma \vdash \varphi$;

Cut: if $\Gamma \vdash \varphi$ for all $\varphi \in \Phi$, and $\Phi \vdash \psi$ then $\Gamma \vdash \psi$;

Weakening: if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Phi$ then $\Phi \vdash \varphi$.

We will consider only these three conditions, though more conditions could be imposed. For example, in [BP89], Blok and Pigozzi considered logics that further satisfy the *finitariness* condition:

Finitariness: If $\Gamma \vdash \varphi$ then $\Gamma' \vdash \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

When considering several logics, to avoid confusion, we shall sometimes write $\vdash_{\mathcal{L}}$ instead of $\vdash$. In the sequel if $\Gamma, \Phi \subseteq L$, we shall write $\Gamma \vdash \Phi$ whenever $\Gamma \vdash \varphi$ for all $\varphi \in \Phi$. We say that φ and ψ are *interderivable*, which is denoted by $\varphi \dashv\vdash \psi$, if $\varphi \vdash \psi$ and $\psi \vdash \varphi$. In the same way, given $\Gamma, \Phi \subseteq L$ we say that Γ and Φ are *interderivable*, if $\Gamma \vdash \Phi$ and $\Phi \vdash \Gamma$.

It is well known that Reflexivity and Cut together imply Weakening:
Let $\Gamma \subseteq \Phi \subseteq L$ and $\varphi \in L$. Suppose also that $\Gamma \vdash \varphi$. We want to prove that $\Phi \vdash \varphi$. Let $\psi \in \Gamma$, then, because $\Gamma \subseteq \Phi$, we have that $\psi \in \Phi$. By reflexivity we know that $\Phi \vdash \psi$ for all $\psi \in \Gamma$. Using the hypothesis that $\Gamma \vdash \varphi$ and using Cut, we conclude that $\Phi \vdash \varphi$.
Despite this fact, we have kept Weakening in the definition for methodological reasons, thus explicitly excluding non-monotonic logics from this context.

In the literature it is usual to define a logic as a pair $\mathcal{L} = \langle L, ^{\vdash} \rangle$ where L is a set and $^{\vdash}$ is a *closure operator* on L (in the sense of Kuratowski [MP96]), that is, $^{\vdash} : 2^L \to 2^L$ is a function that satisfies the following properties:

Extensiveness: $\Gamma \subseteq \Gamma^{\vdash}$;

Monotonicity: If $\Gamma \subseteq \Phi$ then $\Gamma^{\vdash} \subseteq \Phi^{\vdash}$;

Idempotence: $(\Gamma^{\vdash})^{\vdash} \subseteq \Gamma^{\vdash}$.

Let us see that this is an equivalent way to define a logic:

Let $\mathcal{L} = \langle L, \vdash \rangle$ be a logic. Given a set $\Gamma \subseteq L$, we can consider the set $\Gamma^{\vdash} = \{\varphi \in L : \Gamma \vdash \varphi\}$. This function on the powerset of L is called the *consequence operator* of $\mathcal{L}$ and satisfies the Kuratowski axioms.

Extensiveness follows directly from the Reflexivity of $\vdash$. If $\varphi \in \Gamma$ then by Reflexivity $\Gamma \vdash \varphi$ and hence, by definition of $\Gamma^{\vdash}$, $\varphi \in \Gamma^{\vdash}$.

Monotonicity follows from the Weakening property of $\vdash$. Suppose that $\Gamma \subseteq \Phi$ and $\varphi \in \Gamma^{\vdash}$, that is, $\Gamma \vdash \varphi$. By Weakening, $\Phi \vdash \varphi$, and so, $\varphi \in \Phi^{\vdash}$.

Idempotence follows from the Cut property of $\vdash$. Let $\varphi \in (\Gamma^{\vdash})^{\vdash}$, that is, $\Gamma^{\vdash} \vdash \varphi$. We also know that $\Gamma \vdash \psi$ for all $\psi \in \Gamma^{\vdash}$. Hence by Cut we have that $\Gamma \vdash \varphi$ and this means that $\varphi \in \Gamma^{\vdash}$.

Let $\mathcal{L} = \langle L, ^{\vdash} \rangle$ be a logic in the second sense. We now aim to prove that the relation $\vdash \subseteq 2^L \times L$, defined by $\Gamma \vdash \varphi$ iff $\varphi \in \Gamma^{\vdash}$ satisfies Reflexivity, Cut and Weakening.

Reflexivity follows from Extensiveness. If $\varphi \in \Gamma$, then by Extensiveness $\varphi \in \Gamma^{\vdash}$, that is, $\Gamma \vdash \varphi$.

Cut follows from Idempotence and Monotonicity. Suppose that $\Gamma \vdash \varphi$ for all $\varphi \in \Phi$, and $\Phi \vdash \psi$. We want to prove that $\Gamma \vdash \psi$. By hypothesis we know that $\Phi \subseteq \Gamma^{\vdash}$, and then by Monotonicity and Idempotence we have that $\Phi^{\vdash} \subseteq \Gamma^{\vdash}$. Therefore $\psi \in \Gamma^{\vdash}$, that is, $\Gamma \vdash \psi$.

Weakening follows from the Monotonicity of $^{\vdash}$. Suppose that $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Phi$. We want to prove that $\Phi \vdash \varphi$. We know that $\varphi \in \Gamma^{\vdash}$ and so, by Monotonicity, we have that $\varphi \in \Phi^{\vdash}$, that is, $\Phi \vdash \varphi$.

In the sequel we shall use either the consequence relation or the consequence operator, interchangeably, whenever it is more convenient.

The *theorems* of $\mathcal{L}$ are the formulas φ such that $\emptyset \vdash \varphi$. A *theory* of $\mathcal{L}$, or briefly a $\mathcal{L}$*-theory*, is a set Γ of formulas such that Γ is closed under the consequence relation $\vdash$, that is, such that if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$. Given a set Γ, we can consider the set $\Gamma^{\vdash}$, the smallest theory containing Γ. The set of all theories of $\mathcal{L}$ is denoted by $Th_{\mathcal{L}}$.

Definition 2.1.2 (Partial order)
A *partial order* is a pair $\langle R, \leqslant \rangle$, where R is a set and $\leqslant \subseteq R \times R$ is a relation satisfiying the following properties:

Reflexivity: $r \leqslant r$ for all $r \in R$;

Transitivity: $r_1 \leqslant r_2$ and $r_2 \leqslant r_3$ implies $r_1 \leqslant r_3$ for all $r_1, r_2, r_3 \in R$;

Antisymmetry: $r_1 \leqslant r_2$ and $r_2 \leqslant r_1$ implies $r_1 = r_2$ for all $r_1, r_2 \in R$.

If $\langle P, \leqslant \rangle$ is a partial order, then P is called a *partial ordered set* or simply a *poset*. Let A be a subset of a poset P. An element $p \in P$ is an *upper bound* for A if $a \leqslant p$ for every a in A. An element $p \in P$ is the *least upper bound of* A, or *supremum* of A ($\bigvee A$) if p is an upper bound of A, and $a \leqslant b$ for every $a \in A$ implies $p \leqslant b$ (i.e., p is the smallest among the upper bounds of A). Similarly we can define what it means for p to be a *lower bound of* A, and for p to be the *greatest lower bound of* A, also called the *infimum* of A ($\bigwedge A$).

Definition 2.1.3 (Complete partial order)
A poset P is *complete* if for every subset A of P both $\bigvee A$ and $\bigwedge A$ exist (in P).

Definition 2.1.4 (Maps of partial orders)
Given two partial orders $\langle A, \leqslant_A \rangle$ and $\langle B, \leqslant_B \rangle$, a *map from* $\langle A, \leqslant_A \rangle$ *to* $\langle B, \leqslant_B \rangle$ is a function $h : A \rightarrow B$.

The map h is *monotone* if for all $a_1, a_2 \in A$,

$$a_1 \leqslant_A a_2 \text{ implies } h(a_1) \leqslant_B h(a_2).$$

The map h is *sup-preserving* if for all $A_1 \subseteq A$

$$h(\bigvee A_1) = \bigvee h[A_1].$$

It is easy to see that a sup-preserving map is monotone. To prove this, suppose $h : \langle A, \leqslant_A\rangle \to \langle B, \leqslant_B\rangle$ is a sup-preserving map and consider $a_1, a_2 \in A$ such that $a_1 \leqslant_A a_2$. Then $\bigvee\{a_1, a_2\} = a_2$. Because h is sup-preserving we have that $h(\bigvee\{a_1, a_2\}) = \bigvee h[\{a_1, a_2\}]$, which means that $h(a_2) = \bigvee\{h(a_1), h(a_2)\}$, and so $h(a_1) \leqslant_B h(a_2)$.

We now want to see that $\langle Th_{\mathcal{L}}, \subseteq\rangle$ forms a complete partial order, where the supremum and the infimum of a set T of theories are respectively, $\bigvee^{\mathcal{L}} T = (\cup_{\Gamma\in T}\Gamma)^{\vdash}$ and $\bigwedge^{\mathcal{L}} T = \bigcap_{\Gamma\in T}\Gamma$:

By the properties of $\subseteq$ is trivial to verify that $\langle Th_{\mathcal{L}}, \subseteq\rangle$ forms a partial order. Hence, all we need to prove is that this is a complete partial order, that is, for every $T \subseteq Th_{\mathcal{L}}$ the supremum and the infimum of T exists in $Th_{\mathcal{L}}$. For this we will first prove that $\bigvee^{\mathcal{L}} T$ and $\bigwedge^{\mathcal{L}} T$ are well defined, that is, they are still in $Th_{\mathcal{L}}$. By definition, $\bigvee^{\mathcal{L}} T$ is always a theory, and hence belongs to $Th_{\mathcal{L}}$. We now prove that the arbitrary intersection of theories is still a theory. Let $I = \bigcap_{\Gamma\in T}\Gamma$. If $I \vdash \varphi$ then by monotonicity $\Gamma \vdash \varphi$ for every $\Gamma \in T$. Because each $\Gamma \in T$ is a theory, $\varphi \in \Gamma$ for every $\Gamma \in T$, and hence $\varphi \in I$. We conclude that $I \in Th_{\mathcal{L}}$. We will now prove that in fact $\bigvee^{\mathcal{L}} T$ and $\bigwedge^{\mathcal{L}} T$ are, respectively, the supremum and the infimum of T.

By definition, $\bigvee^{\mathcal{L}} T$ is clearly a theory that contains all $\Gamma \in T$. Let $\Phi \in Th_{\mathcal{L}}$ such that $\bigcup_{\Gamma\in T}\Gamma \subseteq \Phi$, then by Monotonicity, $(\bigcup_{\Gamma\in T}\Gamma)^{\vdash} \subseteq (\Phi)^{\vdash} = \Phi$.

By definition, $\bigwedge^{\mathcal{L}} T$ it is clear that $\bigwedge^{\mathcal{L}} T \subseteq \Gamma$ for all $\Gamma \in T$. Let $\Phi \in Th_{\mathcal{L}}$ such that $\Phi \subseteq \Gamma$ for all $\Gamma \in T$. Then trivially $\Phi \subseteq \bigcap_{\Gamma\in T}\Gamma = \bigwedge^{\mathcal{L}} T$.

When we want to introduce a particular logic, it is useful to introduce it as a *deductive system*.

Definition 2.1.5 (Deductive system)
A *deductive system* is a pair $\mathcal{D} = \langle L, R\rangle$ where L is a set (of *formulas*), and R is a subset of $(\wp_{fin}L) \times L$.

Each element $r = \langle\Gamma, \varphi\rangle$ of R is called an *inference rule*. We say that the (finite) set Γ is the set of *premises* of r, which we will denote by $Prem(r)$, and that φ is the *conclusion* of r, which we denote by $Conc(r)$. If $Prem(r) = \emptyset$, r is said to be an *axiom*, as well as $Conc(r)$.

Given a deductive system it is easy to see what logic is *naturally* associated with it.

Definition 2.1.6 (Derivability)
Given a *deductive system* $\mathcal{D} = \langle L, R\rangle$, a formula $\varphi \in L$ is *derivable from a set of formulas* $\Gamma \subseteq L$ in $\mathcal{D}$, denoted by $\Gamma \vdash_{\mathcal{D}} \varphi$, if there exists a sequence $\gamma_1, \ldots, \gamma_m \in L$ such that:

- γ_m is φ;

- for each $i = 1, \ldots, m$, the formula γ_i is either:
  - an element of Γ, or
  - there exists a rule $r \in R$ such that $\gamma_i = Conc(r)$ and $Prem(r) \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

The logic associated with $\mathcal{D}$ is $\mathcal{L}_{\mathcal{D}} = \langle L, \vdash_{\mathcal{D}} \rangle$. Note that, since all rules have finite sets of premises, the logic $\mathcal{L}_{\mathcal{D}}$ is always finitary.

2.2 Maps of logics

Let $\mathcal{L} = \langle L, \vdash \rangle$ and $\mathcal{L}' = \langle L', \vdash' \rangle$ be two logics.

Definition 2.2.1 (Map of logics)
A *map* θ from $\mathcal{L}$ to $\mathcal{L}'$ is a function $\theta : L \to 2^{L'}$ such that the following holds:

$$\text{if } \Gamma \vdash \varphi \text{ then } (\bigcup_{\gamma \in \Gamma} \theta(\gamma)) \vdash' \theta(\varphi).$$

The cases where $\theta(\varphi)$ is a singleton set for every $\varphi \in L$ or is a finite set for every $\varphi \in L$, are usual particular cases of the above definition of map. For the sake of notation we will use $\theta[\Gamma] = \bigcup_{\gamma \in \Gamma} \theta(\gamma)$. Using this notation the condition of a map can be rewritten to:

$$\text{if } \Gamma \vdash \varphi \text{ then } \theta[\Gamma] \vdash' \theta(\varphi).$$

Definition 2.2.2 (Conservative Map)
A map θ from $\mathcal{L}$ to $\mathcal{L}'$ is *conservative* when

$$\Gamma \vdash \varphi \quad \text{iff} \quad \theta[\Gamma] \vdash' \theta(\varphi).$$

Definition 2.2.3 (Strong Representation)
A *strong representation* of $\mathcal{L}$ in $\mathcal{L}'$ is a pair (θ, τ) of conservative maps $\theta : \mathcal{L} \to \mathcal{L}'$ and $\tau : \mathcal{L}' \to \mathcal{L}$ such that:

i) For all $\varphi \in L$ we have that $\varphi \dashv\vdash \tau[\theta(\varphi)]$

ii) For all $\varphi' \in L'$ we have that $\varphi' \dashv\vdash' \theta[\tau(\varphi')]$;

In fact, in the above definition, one of the conditions **i)** or **ii)** is enough, since, given the conservativeness of the maps θ and τ, **i)** is equivalent to **ii)**.

> Suppose, without loss of generality, that we have **ii)** and consider $\varphi \in L$. Then, because $\theta(\varphi) \subseteq L'$, by **ii)**, we have that for all $\psi \in \theta(\varphi)$, $\psi \dashv\vdash' \theta[\tau(\psi)]$. Then $\theta(\varphi) \dashv\vdash' \theta[\tau[\theta(\varphi)]]$. By conservativeness of θ we have $\varphi \dashv\vdash \tau[\theta(\varphi)]$.

In fact if we assume the conservativeness of θ and consider any function $\tau : L' \to 2^L$ that satisfies **ii)**, then we can conclude that τ is in fact a conservative map from $\mathcal{L}'$ to $\mathcal{L}$ that also satisfies **i)**.

> Suppose $\theta : \mathcal{L} \to \mathcal{L}'$ is a conservative map. Let $\tau : L' \to 2^L$ be any function that satisfies **ii)**. We aim to prove that τ is a conservative map that satisfies **i)**. First we will prove that τ is conservative, that is, for all $\Gamma' \cup \{\varphi'\} \subseteq L'$ we have that $\Gamma' \vdash' \varphi'$ iff $\tau[\Gamma'] \vdash \tau(\varphi')$.
>
> > Suppose first that $\Gamma' \vdash' \varphi'$. By **ii)** we have $\theta[\tau[\Gamma']] \vdash' \theta[\tau(\varphi')]$ and by the conservativeness of θ we get that $\tau[\Gamma'] \vdash \tau(\varphi')$.
> >
> > Suppose now that $\tau[\Gamma'] \vdash \tau(\varphi')$. Because θ is conservative we get that $\theta[\tau[\Gamma']] \vdash' \theta[\tau(\varphi')]$. Using **ii)** we get that $\Gamma' \vdash' \varphi'$.
>
> We proved above that, assuming the conservativeness of θ and τ, condition **i)** is equivalent to **ii)**. In this case we know that both θ and τ are conservative, and since we are assuming **ii)** we also have **i)**.

Note also that, because of the symmetry of the definition, we also have a strong representation (τ, θ) of $\mathcal{L}'$ in $\mathcal{L}$.

Given a map $\theta : \mathcal{L} \to \mathcal{L}'$ we can consider the function between complete partial orders $\theta^{Th} : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$, such that $\theta^{Th}(\Gamma) = \theta[\Gamma]^{\vdash'}$. We can also consider the function between complete partial orders $\theta^{-1} : Th_{\mathcal{L}'} \to Th_{\mathcal{L}}$ such that $\theta^{-1}(\Delta) = \{\varphi \in L : \theta(\varphi) \subseteq \Delta\}$. Let us prove that θ^{-1} is well-defined, that is, that $\theta^{-1}(\Delta)$ is a $\mathcal{L}$-theory for all $\mathcal{L}'$-theory Δ. Suppose that $\theta^{-1}(\Delta) \vdash \varphi$. We want to see that $\varphi \in \theta^{-1}(\Delta)$. Since θ is a map we have that $\theta[\theta^{-1}(\Delta)] \vdash' \theta(\varphi)$. Since $\theta[\theta^{-1}(\Delta)] \subseteq \Delta$, by monotonicity we get that $\Delta \vdash' \theta(\varphi)$. Then $\theta(\varphi) \subseteq \Delta$ since Δ is a $\mathcal{L}'$-theory. By definition of $\theta^{-1}(\Delta)$, $\varphi \in \theta^{-1}(\Delta)$.

Proposition 2.2.4
Suppose $\theta : \mathcal{L} \to \mathcal{L}'$ is a map. Then $\theta^{Th} : Th_{\mathcal{L}} \to Th_{\mathcal{L}'}$ is sup-preserving.

Proof: Let $T \subseteq Th_{\mathcal{L}}$. Then

$$\begin{aligned}\theta^{Th}(\textstyle\bigvee^{\mathcal{L}} T) &= \theta^{Th}((\textstyle\bigcup_{\Gamma\in T}\Gamma)^{\vdash}) &&= (\theta[(\textstyle\bigcup_{\Gamma\in T}\Gamma)^{\vdash}])^{\vdash'} \\ &= (\textstyle\bigcup_{\Gamma\in T}\theta[\Gamma]^{\vdash'})^{\vdash'} &&= (\textstyle\bigcup_{\Gamma\in T}\theta^{Th}(\Gamma))^{\vdash'} \\ &= \textstyle\bigvee^{\mathcal{L}'}\theta^{Th}[T]\end{aligned}$$

QED

Proposition 2.2.5
Suppose $\theta : \mathcal{L} \to \mathcal{L}'$ is a map. Then $\theta^{-1} : Th_{\mathcal{L}'} \to Th_{\mathcal{L}}$ is monotone.

Proof: Let $T_1, T_2 \subseteq Th_{\mathcal{L}'}$ such that $T_1 \subseteq T_2$. We want to show that $\theta^{-1}[T_1] \subseteq \theta^{-1}[T_2]$. Let $\varphi \in \theta^{-1}[T_1]$, that is, $\varphi \in \bigcup_{\Delta_1 \in T_1} \theta^{-1}[\Delta_1]$. Then because $T_1 \subseteq T_2$, $\varphi \in \bigcup_{\Delta_2 \in T_2} \theta^{-1}[\Delta_2]$, that is, $\varphi \in \theta^{-1}[T_2]$.

QED

Proposition 2.2.6
Suppose $\theta : \mathcal{L} \to \mathcal{L}'$ is a map. Then for all $\mathcal{L}$-theory Γ and all $\mathcal{L}'$-theory Δ we have that:

$$\theta^{-1}(\theta^{Th}(\Gamma)) \supseteq \Gamma$$

$$\theta^{Th}(\theta^{-1}(\Delta)) \subseteq \Delta$$

Proof: We will first prove that $\theta^{-1}(\theta^{Th}(\Gamma)) \supseteq \Gamma$.

$$\begin{aligned} \theta^{-1}(\theta^{Th}(\Gamma)) &= \{\varphi \in L : \theta(\varphi) \subseteq \theta^{Th}(\Gamma)\} \\ &= \{\varphi \in L : \theta(\varphi) \subseteq \theta[\Gamma]^{\vdash'}\} \\ &= \{\varphi \in L : \theta[\Gamma] \vdash' \theta(\varphi)\} \end{aligned}$$

Now it is easy to see that $\theta^{-1}(\theta^{Th}(\Gamma)) \supseteq \Gamma$.

Let us now prove that $\theta^{Th}(\theta^{-1}(\Delta)) \subseteq \Delta$.

$$\begin{aligned} \theta^{Th}(\theta^{-1}(\Delta)) &= \theta^{Th}(\{\varphi \in L : \theta(\varphi) \subseteq \Delta\}) \\ &= (\textstyle\bigcup_{\varphi \in \theta^{-1}(\Delta)} \theta(\varphi))^{\vdash'} \\ &\subseteq \Delta^{\vdash'} \\ &= \Delta \end{aligned}$$

QED

Proposition 2.2.7
Suppose $\theta : \mathcal{L} \to \mathcal{L}'$ is a conservative map. Then

$$\theta^{-1}(\theta^{Th}(\Gamma)) = \Gamma$$

Proof: Let us prove the two inclusions. We already proved that $\theta^{-1}(\theta^{Th}(\Gamma)) \supseteq \Gamma$. To prove that $\theta^{-1}(\theta^{Th}(\Gamma)) \subseteq \Gamma$ consider $\varphi \in \theta^{-1}(\theta^{Th}(\Gamma))$. As we have already saw $\theta^{-1}(\theta^{Th}(\Gamma)) = \{\varphi \in L : \theta[\Gamma] \vdash' \theta(\varphi)\}$, so $\theta[\Gamma] \vdash' \theta(\varphi)$. Then, by the conservativeness of θ, we get that $\Gamma \vdash \varphi$. Since Γ is a theory, we have that $\varphi \in \Gamma$.

QED

Proposition 2.2.8
Suppose (θ, τ) is a strong representation of the logic $\mathcal{L}$ in the logic $\mathcal{L}'$. Then we have the following:

$$\theta^{-1} = \tau^{Th}$$

$\theta^{Th} = \tau^{-1}$

Proof: Since (θ, τ) is a strong representation, then θ and τ are both conservative maps and for all $\varphi \in L$ we have that $\varphi \dashv\vdash \tau[\theta(\varphi)]$ and for all $\varphi' \in L'$ we have that $\varphi' \dashv\vdash' \theta[\tau(\varphi')]$. First let us prove that $\theta^{-1} = \tau^{Th}$. For this we will prove that, for all $\mathcal{L}'$-theory Δ, $\theta^{-1}(\Delta) \subseteq \tau^{Th}(\Delta)$ and $\theta^{-1}(\Delta) \supseteq \tau^{Th}(\Delta)$. Let $\varphi \in \theta^{-1}(\Delta)$, then $\theta(\varphi) \subseteq \Delta$. By reflexivity we get that $\Delta \vdash' \theta(\varphi)$ and, since τ is a map, we get that $\tau[\Delta] \vdash \tau[\theta(\varphi)]$. By hypothesis we know that $\tau[\theta(\varphi)] \vdash \varphi$ and then by cut we have that $\tau[\Delta] \vdash \varphi$, that is, $\varphi \in \tau[\Delta]^{\vdash}$ which means that $\varphi \in \tau^{Th}(\Delta)$. We then conclude that $\theta^{-1}(\Delta) \subseteq \tau^{Th}(\Delta)$.

Let $\varphi \in \tau^{Th}(\Delta)$, that is, $\varphi \in \tau[\Delta]^{\vdash}$. Then $\tau[\Delta] \vdash \varphi$. By hypothesis $\varphi \vdash \tau[\theta(\varphi)]$. Then, using cut, we have that $\tau[\Delta] \vdash \tau[\theta(\varphi)]$. Using the conservativeness of τ, we have $\Delta \vdash' \theta(\varphi)$. Since Δ is a theory, we conclude that $\theta(\varphi) \subseteq \Delta$, that is, $\varphi \in \theta^{-1}(\Delta)$.

The proof of the equality $\theta^{Th} = \tau^{-1}$ is analogue. QED

We get an immediate corollary of the above proposition that will be useful later.

Corollary 2.2.9
Suppose (θ, τ) is a strong representation of the logic $\mathcal{L}$ in the logic $\mathcal{L}'$. Then θ^{Th} is a bijection.

2.3 Structural propositional logics

A *propositional signature* is an indexed set $\Sigma = \{\Sigma_i\}_{i \in \mathbb{N}_0}$ where each Σ_i is the set of i-ary constructors. We consider fixed a set $\Xi = \{\xi_i\}_{i \in \mathbb{N}}$ of *propositional variables*. Given a signature Σ, the *language over* Σ, which we denote by L_Σ, is build inductively in the usual way from Σ over the set Ξ:

- $\Xi \cup \Sigma_0 \subseteq L_\Sigma$;
- If $\varphi_1, \ldots, \varphi_n \in L_\Sigma$ and $c \in \Sigma_n$ then $c(\varphi_1, \ldots, \varphi_n) \in L_\Sigma$.

We call Σ-*formulas* to the elements of L_Σ , or simply *formulas* when Σ is clear from the context. Note that L_Σ is precisely the carrier of the free Σ-algebra, $\mathbf{L}_{\boldsymbol{\Sigma}}(\boldsymbol{\Xi})$, that has as generators the elements of Ξ and is usually called the *algebra of formulas*. For simplicity we usually write $\mathbf{L}_{\boldsymbol{\Sigma}}$ instead of $\mathbf{L}_{\boldsymbol{\Sigma}}(\boldsymbol{\Xi})$.

Given $\Xi' \subseteq \Xi$, we will consider the set $L_\Sigma(\Xi') \subseteq L_\Sigma$ of all Σ-formulas generated from Ξ'. Of course $L_\Sigma(\Xi) = L_\Sigma$.

Given a signature Σ, a *substitution* is a function $\sigma : \Xi \to L_\Sigma$. This function extends to a unique endomorphism of the formula algebra, which we will denote also by σ, by requiring that:

- $\sigma(c) = c$ for all $c \in \Sigma_0$;
- $\sigma(c(\varphi_1, \ldots, \varphi_n)) = c(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ for all $c \in \Sigma_n$ and all $\varphi_1, \ldots, \varphi_n \in L_\Sigma$.

Note that this is precisely the free extension of σ to $\mathbf{L}_{\boldsymbol{\Sigma}}$. Given a set $\Gamma \subseteq L_\Sigma$ we can also consider the set $\sigma[\Gamma] = \{\sigma(\varphi) : \varphi \in \Gamma\}$.

Given a formula $\varphi \in L_\Sigma(\{\xi_1, \ldots, \xi_k\})$ we will write $\varphi(\xi_1 \backslash \psi_1, \ldots, \xi_k \backslash \psi_k)$ to denote the formula $\sigma(\varphi(\xi_1, \ldots, \xi_k))$ whenever $\sigma(\xi_1) = \psi_1, \ldots, \sigma(\xi_k) = \psi_k$.

Definition 2.3.1 (Structural propositional logic)
A *structural propositional logic* is a pair $\mathcal{L} = \langle \Sigma, \vdash \rangle$, where Σ is a propositional signature and $\langle L_\Sigma, \vdash \rangle$ is a logic that also satisfies [Tar83]:

Structurality: if $\Gamma \vdash \varphi$ then $\sigma[\Gamma] \vdash \sigma(\varphi)$ for every substitution σ.

When we are dealing with such structural propositional logics we can particularize the notion of deductive system, but now using schematic rules.

Definition 2.3.2 (Structural deductive system)
A *structural deductive system* is a pair $\mathcal{D} = \langle \Sigma, R \rangle$ where Σ is a propositional signature, and R is a subset of $(\wp_{fin} L_\Sigma) \times L_\Sigma$.

Definition 2.3.3 (Derivability)
Given a structural deductive system $\mathcal{D} = \langle \Sigma, R \rangle$, the consequence relation associated with $\mathcal{D}$, $\vdash_{\mathcal{D}}$, is the one associated with the deductive system $\mathcal{D} = \langle L_\Sigma, \{\sigma(r) : r \in R, \sigma \text{ substitution}\} \rangle$.

The logic associated with $\mathcal{D}$ is $\mathcal{L}_{\mathcal{D}} = \langle L, \vdash_{\mathcal{D}} \rangle$. Note that, since all rules have finite sets of premises, the logic $\mathcal{L}_{\mathcal{D}}$ is always finitary.

2.4 Examples

In this section we will present some well known examples of logics. The first two are examples of propositional structural logics, while the other are not propositional. As expected, we will introduce these examples by means of a deductive system.

Example 2.4.1 Paraconsistent logic $\mathcal{C}_1$ (da Costa, 1963)

- Signature Σ:
  - $C_0 = \{\mathbf{t}, \mathbf{f}\}$;
  - $C_1 = \{\neg\}$;
  - $C_2 = \{\wedge, \vee, \supset\}$;

- $C_i = \varnothing$ for all $i > 2$.

- $P = \{\langle\varnothing, \xi_1 \supset (\xi_2 \supset \xi_1)\rangle,$
$\langle\varnothing, (\xi_1 \supset (\xi_2 \supset \xi_3)) \supset ((\xi_1 \supset \xi_2) \supset (\xi_1 \supset \xi_3))\rangle,$
$\langle\varnothing, (\xi_1 \wedge \xi_2) \supset \xi_1\rangle,$
$\langle\varnothing, (\xi_1 \wedge \xi_2) \supset \xi_2\rangle,$
$\langle\varnothing, \xi_1 \supset (\xi_2 \supset (\xi_1 \wedge \xi_2))\rangle,$
$\langle\varnothing, \xi_1 \supset (\xi_1 \vee \xi_2)\rangle,$
$\langle\varnothing, \xi_2 \supset (\xi_1 \vee \xi_2)\rangle,$
$\langle\varnothing, (\xi_1 \supset \xi_3) \supset ((\xi_2 \supset \xi_3) \supset ((\xi_1 \vee \xi_2) \supset \xi_3))\rangle,$
$\langle\varnothing, \neg\neg\xi_1 \supset \xi_1\rangle,$
$\langle\varnothing, \xi_1 \vee \neg\xi_1\rangle,$
$\langle\varnothing, \xi_1^\circ \supset (\xi_1 \supset (\neg\xi_1 \supset \xi_2))\rangle,$
$\langle\varnothing, (\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \wedge \xi_2)^\circ\rangle,$
$\langle\varnothing, (\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \vee \xi_2)^\circ\rangle,$
$\langle\varnothing, (\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \supset \xi_2)^\circ\rangle,$
$\langle\varnothing, \mathbf{t} \equiv (\xi_1 \supset \xi_1)\rangle,$
$\langle\varnothing, \mathbf{f} \equiv (\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1))\rangle,$
$\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle\}$

where φ° is an abbreviation of $\neg(\varphi \wedge (\neg\varphi))$ and $\varphi \equiv \psi$ is an abbreviation of $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$.

Example 2.4.2 Classical propositional logic CPL

- Signature Σ
  - $C_0 = \varnothing$
  - $C_1 = \{\neg\}$
  - $C_2 = \{\Rightarrow\}$
  - $C_n = \varnothing$, for all $n > 2$

- $P = \{\langle\varnothing, (\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
$\langle\varnothing, ((\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3)))\rangle,$
$\langle\varnothing, ((\neg\xi_1 \Rightarrow \neg\xi_2) \Rightarrow (\xi_2 \Rightarrow \xi_1))\rangle,$
$\langle\{\xi_1, \xi_1 \Rightarrow \xi_2\}, \xi_2\rangle\}$

Before going to the next examples let us introduce some necessary definitions. An *equational signature* is a pair $\Sigma = \langle S, O\rangle$, where S is a set, called the set of *sorts*, and $O = \{O_{ws}\}_{w \in S^*, s \in S}$ is a family of disjoint sets. We say that an equational signature $\Sigma = \langle S, O \rangle$ is n-sorted if $n = |S|$.

Given an equational signature Σ, a Σ-*algebra* is a pair $\mathbf{A}{=}\langle\{A_s\}_{s\in S}, \_{\mathbf{A}}\rangle$, where each A_s is a non-empty set, called the *carrier of sort* s, and $\_{A}$ assigns to each operation $o \in O_{s_1 \ldots s_n s}$ a function $\underline{o}_A : A_{s_1} \times \ldots \times A_{s_n} \rightarrow A_s$. The set of all Σ-algebras will be denoted by Alg_Σ.

A *homomorphism* from the Σ-algebra $\mathbf{A}$ to the Σ-algebra $\mathbf{B}$, $h : \mathbf{A} \rightarrow \mathbf{B}$, is a set $\{h_s : A_s \rightarrow B_s\}_{s \in S}$, such that for all $o \in O_{s_1 \ldots s_n s}$, we have that $h_s(\underline{o}_A(a_1, \ldots, a_n)) = \underline{o}_B(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$.

A *congruence* on a Σ-algebra $\mathbf{A}$ is a set $\equiv\, = \{\equiv_s\}_{s\in S}$, such that $\equiv_s$ is an equivalence relation on A_s and for each $o \in O_{s_1 \ldots s_n s}$, we have that, if $a_1 \equiv_{s_1} b_1$ $, \ldots, a_n \equiv_{s_n} b_n$ then $\underline{o}_{\mathbf{A}}(a_1, \ldots, a_n) \equiv_s \underline{o}_{\mathbf{A}}(b_1, \ldots, b_n)$. We denote by $Cong_{\mathbf{A}}$ the set of all congruences on the algebra $\mathbf{A}$. Given $\theta_1, \theta_2 \in Cong_{\mathbf{A}}$, we can define the operation $\circ : Cong_{\mathbf{A}} \times Cong_{\mathbf{A}} \rightarrow Cong_{\mathbf{A}}$, where for all sort s, $\langle a, b \rangle \in (\theta_1 \circ \theta_2)_s$ iff there exists a $c \in A$ such that $\langle a, c \rangle \in (\theta_1)_s$ and $\langle c, b \rangle \in (\theta_2)_s$. Inductively one defines $r_1 \circ r_2 \circ \ldots \circ r_n = (r_1 \circ r_2 \circ \ldots \circ r_{n-1}) \circ r_n$. It is not difficult to see that $\langle Congr_A, \subseteq \rangle$ is a complete partial order, where for $\{\theta_i\}_{i\in I} \subseteq Congr_A$, $\bigwedge\{\theta_i\}_{i \in I} = \bigcap_{i\in I} \theta_i$ and $\bigvee\{\theta_i\}_{i\in I} = \bigcup\{\theta_{i_1} \circ \theta_{i_2} \circ \ldots \circ \theta_{i_k} : i_1, i_2, \ldots, i_k \in I, k < \infty\}$.

Given an equational signature $\Sigma = \langle S, O \rangle$, we will consider fixed a family $X = \{X_s\}_{s \in S}$ of disjoint sets, where X_s is the set of *variables* of sort s. We denote by $\mathbf{T(\Sigma, X)}$ the free Σ-algebra with generators X. Each set $T(\Sigma, X)_s$ is called the set of *terms* over (Σ, X) of sort s. Given $X' \subseteq X$ we can consider $\mathbf{T(\Sigma, X')}$ the free Σ-algebra generated by X'. We denote by $gT(\Sigma)$ the family $T(\Sigma, \varnothing)$. Each set $gT(\Sigma)_s$ is called the set of *ground terms* over Σ of sort s.

We also consider $Eq(\Sigma, X)$, the set of *equations* over (Σ, X), defined by $Eq(\Sigma, X) = \{t_1 \approx t_2 : t_1, t_2 \in T(\Sigma, X)_s \text{ for some s} \in S\}$. Each set $Eq(\Sigma, X)_s = \{t_1 \approx t_2 : t_1, t_2 \in T(\Sigma, X)_s\}$ is called the set of equations over (Σ, X) of sort s. In the same way as above we can consider the set of *ground equations*, $gEq(\Sigma) = Eq(\Sigma, \varnothing)$. The set $gEq(\Sigma)_s = Eq(\Sigma, \varnothing)_s$ denote the set of ground equations of sort s.

By an *assignment* of X over a Σ-algebra $\mathbf{A}$ we mean a family $\rho = \{\rho_s\}_{s \in S}$ such that $\rho_s : X_s \rightarrow A_s$. Given a assignment ρ of X over $\mathbf{A}$, the *denotation of terms* is the homomorphism $[\![\_]\!]^\rho_{\mathbf{A}} : \mathbf{T(\Sigma, X)} \rightarrow \mathbf{A}$ such that, for all $s \in S$ and all $x \in X_s$, $[\![x]\!]^\rho_{\mathbf{A}} = \rho_s(x)$. Note that, since $\mathbf{T(\Sigma)}$ is the free Σ-algebra with generators X, the value of a term over a homomorphism h depends only on the value h assigns to the variables.

Given a Σ-algebra $\mathbf{A}$, an assignment ρ and $t_1 \approx t_2 \in Eq(\Sigma, X)$, we write $\mathbf{A}, \rho \Vdash t_1 \approx t_2$ if $[\![t_1]\!]^\rho_{\mathbf{A}} = [\![t_2]\!]^\rho_{\mathbf{A}}$. We also write $\mathbf{A} \Vdash t_1 \approx t_2$ if, for every assignment ρ, we have that $[\![t_1]\!]^\rho_{\mathbf{A}} = [\![t_2]\!]^\rho_{\mathbf{A}}$, and in this case we say that $\mathbf{A}$ *satisfy* $t_1 \approx t_2$.

Given a class K of Σ-algebras, we write $\Gamma \vDash^\Sigma_K t_1 \approx t_2$ if for all $\mathbf{A} \in K$, if $\mathbf{A} \Vdash \gamma_1 \approx \gamma_2$ for all $\gamma_1 \approx \gamma_2 \in \Gamma$ then $\mathbf{A} \Vdash t_1 \approx t_2$.

It is well known that $\vDash^\Sigma_{Alg_\Sigma}$ coincide [EM85] with the consequence relation associated with the deductive system defined by the rules

reflexivity $\langle \varnothing, t \approx t \rangle$;

symmetry $\langle \{t_1 \approx t_2\}, t_2 \approx t_1 \rangle$;

transitivity $\langle \{t_1 \approx t_2, t_2 \approx t_3\}, t_1 \approx t_3 \rangle$;

congruence $\langle \{t_{11} \approx t_{21}, \dots, t_{1n} \approx t_{2n}\}, o(t_{11}, \dots, t_{1n}) \approx o(t_{21}, \dots, t_{2n}) \rangle$;

substitution $\langle \{t_1 \approx t_2\}, \sigma(t_1) \approx \sigma(t_2) \rangle$.

Example 2.4.3 Unsorted equational logic

Given an 1-sorted equational signature Σ^1 (that can trivially be seen as an unsorted signature as it was introduced in Chapter 2.3) and given a class K of Σ-algebras, we define

$$Eqn^\Sigma_K = \langle gEq(\Sigma), \vDash^\Sigma_K \rangle.$$

Note that, if K is precisely the class of all Σ-algebras that satisfy a given set Δ of equations (with variables), then $\vDash^\Sigma_K$ also coincides with the deductive system obtained by adding the equations in Δ as axioms to the set of rules defined above for the case of $\vDash^\Sigma_{Alg_\Sigma}$.

Example 2.4.4 Two-sorted Equational Logic

Given the 1-sorted equational signatures $\Sigma^{\phi,1}$ and $\Sigma^{\tau,1}$ we can consider the induced 2-sorted equational signature $\Sigma^{\phi,\tau} = \langle S, O \rangle$ where

- $S = \{\phi, \tau\}$, where ϕ is the sort of formulas and τ is the sort of truth-values.
- $O = \{O_{ws}\}_{w \in S^*, s \in S}$, such that:
  - $O_{\epsilon\phi} = \Sigma^{\phi,1}_0 \cup P$;
  - $O_{\phi^k\phi} = \Sigma^{\phi,1}_k$ for $k > 0$;
  - $O_{\phi\tau} = \{v\}$;
  - $O_{\epsilon\tau} = \Sigma^{\tau,1}_0$;
  - $O_{\tau^k\tau} = \Sigma^{\tau,1}_k$ for $k > 0$;

– $O_{ws} = \emptyset$ in the other cases.

Let K be a class of $\Sigma^{\phi,\tau}$-algebras. Then we can define the 2-sorted equational logic

$$Eqn_K^{\Sigma^{\phi,\tau}} = \langle gEq(\Sigma^{\phi,\tau}), \models_K^{\Sigma^{\phi,\tau}} \rangle.$$

Note that, if K is precisely the class of all $\Sigma^{\phi,\tau}$-algebras that satisfy a given set Δ of equations (with variables), then $\models_K^{\Sigma^{\phi,\tau}}$ also coincides with the deductive system obtained by adding the equations in Δ as axioms to the set of rules defined above for the case of $\models_{Alg_\Sigma^{\phi,\tau}}^{\Sigma^{\phi,\tau}}$.

Chapter 3

Algebraization

In this chapter we will recall the abstract theory of algebraization of logics, first introduced in a mathematical precise definition by Blok and Pigozzi in [BP89]. The idea of Blok and Pigozzi was to extend the classical theory of Lindenbaum-Tarski. In [Tar83] Tarski gave the precise connection between classical propositional logic and Boolean algebras. The technique consists on looking at the set of formulas as an algebra with operators induced by the connectives. Logical equivalence is a congruence in the formula algebra and the induced quotient algebra turns out to be a Boolean algebra. This is the so called Lindenbaum - Tarski method.

The definition proposed by Blok and Pigozzi of *algebraizable logic*, is actually what is now called a finitely algebraizable logic [FJP03]. Moreover they considered exclusively finitary logics, that is, logics that also satisfy the finitariness property.

In the sequel, as already stated, we will *not* restrict ourselves to finitary logics and will consider the wider notion of algebraizable logic proposed in [FJP03].

After introducing the main concepts, we will propose an equivalent definition by means of maps between the target logic and unsorted equational logic [EM85].

3.1 Algebraizable logic

As in [BP89], we will restrict ourselves to the study of the algebraizability of structural propositional logics. Consider fixed an arbitrary propositional signature Σ and a set X of equational variables. In this chapter we will use unsorted equational logics, $Eqn^{\Sigma}_{K} = \langle gEq(\Sigma), \vDash^{\Sigma}_{K}\rangle$, as they were introduced in Section 2.4. Before going to the main definitions, let us introduce some notation.

Given a set of equations $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ where $\delta_j, \lambda_j \in T(\Sigma, \{x\})$, it will be useful to consider the following sets $\Theta(x\backslash\varphi) = \{\delta_j(x\backslash\varphi) \approx \lambda_j(x\backslash\varphi) : j \in J\}$ and $\Theta[\Gamma] = \{\delta_j(x\backslash\psi) \approx \lambda_j(x\backslash\psi) : j \in J, \psi \in \Gamma\}$.

Given a set of formulas $E = \{\epsilon_i : i \in I\} \subseteq L_{\Sigma}(\{\xi_1, \xi_2\})$ and a set of equations $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ and an equation $\delta \approx \lambda$ where $\delta_j, \delta, \lambda_j, \lambda \in T(\Sigma, \{x\})$ we can consider the following sets $E(\xi_1\backslash\delta,\ \xi_2\backslash\lambda) = \{\epsilon_i(\xi_1\backslash\delta,\ \xi_2\backslash\lambda) : i \in I\}$ and $E[\Theta] = \{\epsilon_i(\xi_1\backslash\delta_j,\ \xi_2\backslash\lambda_j) : i \in I, j \in J\}$

Definition 3.1.1 (Algebraic Semantics)
Let $\mathcal{L} = \langle\Sigma, \vdash\rangle$ be a structural propositional logic and let K be a class of Σ-algebras. The class K is an *algebraic semantics for* $\mathcal{L}$ if there exists a set of equations $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ where $\delta_j, \lambda_j \in T(\Sigma, \{x\})$, called the set of *defining equations*, such that for every $\Gamma \subseteq L_\Sigma$ and every $\varphi \in L_\Sigma$,

$$\Gamma \vdash \varphi \text{ iff } \Theta[\Gamma] \vDash_K^\Sigma \Theta(x\backslash\varphi).$$

Let us recall the case of classical propositional logic *CPL*.

Example 3.1.2 *The two-element Boolean algebra*

$$\boldsymbol{B} = \langle\{\top, \bot\}, \wedge, \vee, \neg, \top, \bot\rangle$$

constitutes by itself an algebraic semantics for CPL. We just need to consider the singleton set with the equation $x \approx \top$. In fact we have that,

$$\Gamma \vdash_{CPL} \varphi \quad \textit{iff} \quad \{\psi \approx \top : \psi \in \Gamma\} \vDash_{\boldsymbol{B}} \varphi \approx \top$$

A congruence $\equiv$ in a Σ-algebra $\mathbf{A}$ is *compatible* with a subset F of A if whenever $a \in F$ and $\langle a, b\rangle \in \equiv$ then $b \in F$. In this case, F is a union of equivalence classes of $\equiv$.

Definition 3.1.3 (Leibniz operator)
Let $\mathcal{L} = \langle\Sigma, \vdash\rangle$ be a structural propositional logic. Then the *Leibniz operator* on the formula algebra can be given by:

$$\Omega : Th_{\mathcal{L}} \to \mathrm{Congr}_{\mathbf{L_\Sigma}}$$

$$\Gamma \mapsto \text{ largest congruence compatible with } \Gamma.$$

We still have to see that, for all $\mathcal{L}$-theory, $\Omega(\Gamma)$ is well defined, that is, there exits the largest congruence compatible with Γ.

Proposition 3.1.4
The Leibniz operator Ω can be given as

$$\Omega(\Gamma) = \{\langle\varphi, \psi\rangle : \textit{ for all } \delta \in L_\Sigma(\{\xi\}),\ \delta(\xi\backslash\varphi) \in \Gamma \textit{ iff } \delta(\xi\backslash\psi) \in \Gamma\}.$$

Proof: Let $\Theta(\Gamma) = \{\langle\varphi, \psi\rangle : \text{ for all } \delta \in L_\Sigma(\{\xi\}),\ \delta(\xi\backslash\varphi) \in \Gamma \text{ iff } \delta(\xi\backslash\psi) \in \Gamma\}$. We aim to prove that $\Theta(\Gamma) = \Omega(\Gamma)$, that is, $\Theta(\Gamma)$ is the largest congruence compatible with Γ. $\Theta(\Gamma)$ is clearly a congruence and compatible with Γ. We still have to prove that it is the largest one. Let Θ' be a congruence with this property and let $\delta \in L_\Sigma(\{\xi\})$. Since Θ' is a congruence we have, for every $\langle\varphi, \psi\rangle \in \Theta'$, that

$$\langle\delta(\xi\backslash\varphi), \delta(\xi\backslash\psi)\rangle \in \Theta'.$$

Thus by the compatibility of Θ',

$$\delta(\xi\backslash\varphi) \in \Gamma \text{ iff } \delta(\xi\backslash\psi) \in \Gamma.$$

Hence $\langle\varphi, \psi\rangle \in \Theta(\Gamma)$, and so $\Theta' \subseteq \Theta(\Gamma)$.

QED

In fact, this last characterization is also used as the definition of the Leibniz operator. Nevertheless, the first characterization is the most useful for us, in particular because we have in view a generalization of the notion. As we shall see, the Leibniz operator is one of the most important tools in the theory of algebraization. Indeed in [BP89], Blok and Pigozzi use Ω to give intrinsic characterizations of algebraizable logics. Let us see how Ω can be used to define another important class of logics, the *protoalgebraic logics*.

Definition 3.1.5 (Protoalgebraic logic)
A structural propositional logic $\mathcal{L} = \langle\Sigma, \vdash\rangle$ is *protoalgebraic* if, for every theory Γ,

$$\text{if } \langle\varphi, \psi\rangle \in \Omega(\Gamma) \text{ then } \Gamma \cup \{\varphi\} \dashv\vdash \Gamma \cup \{\psi\}.$$

Although in general not algebraizable, the protoalgebraic logics constitute the main class of logics for which the advanced methods of algebraic logic can be applied. In the next proposition we begin to notice the importance of the Leibniz operator, and how we can get important results just by looking at the behavior of Ω on the lattice of $\mathcal{L}$-theories.

Theorem 3.1.6
A structural propositional logic $\mathcal{L} = \langle\Sigma, \vdash\rangle$ is protoalgebraic iff Ω is monotone on the lattice of $\mathcal{L}$-theories.

When we say that a logic $\mathcal{L}$ has an algebraic semantics K, we are saying that, in some fashion the consequence relation of $\mathcal{L}$ can be captured by the equational consequence relation $\vDash_K^\Sigma$. Of course we would be happy if we could also capture, in some interesting way, the equational consequence within the consequence apparatus of $\mathcal{L}$. This is precisely the ideia behind the notion of *algebraizable logic*.

Definition 3.1.7 (Algebraizable Logic)
A structural propositional logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *algebraizable* if there exists a class K of Σ-algebras such that K is an algebraic semantics for $\mathcal{L}$ with defining equations $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ where $\delta_j, \lambda_j \in T(\Sigma, \{x\})$ and there also exists a set of formulas $E = \{\epsilon_i : i \in I\} \subseteq L_\Sigma(\{\xi_1, \xi_2\})$, called the set of *equivalential formulas*, such that the following holds:

- for every set of equations Θ' and every equation $\varphi \approx \psi$ we have
$$\Theta' \vDash_K \varphi \approx \psi \quad \text{iff} \quad E[\Theta'] \vdash E(\xi_1 \backslash \varphi, \xi_2 \backslash \psi);$$
- $\xi \dashv\vdash E[\Theta(x \backslash \xi)]$;
- $x_1 \approx x_2 \;=\!\!\mid\vDash_K \Theta[E(\xi_1 \backslash x_1,\ \xi_2 \backslash x_2)]$.

The definition given by Blok and Pigozzi in [BP89] of algebraizable logic is what is now called *finitely algebraizable logic*, since it assumes that the set E of equivalential formulas and the set Θ of defining equations are finite sets and moreover they consider exclusively finitary logics.

As we said before, the Leibniz operator plays an important role in this theory of algebraization of logics. When Blok and Pigozzi introduced in a precise mathematical sense the notion of algebraizable logic they wanted to give evidence that their definition was the proper one. So they gave some intrinsic natural characterizations of the notion of algebraizable logic that all characterize the same notion. We will present one of those intrinsic characterizations, in which, as already stated, the Leibniz operator is the main tool.

Theorem 3.1.8
A structural propositional logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable iff the Leibniz operator satisfies the following conditions:

i) Ω *is injective;*

ii) Ω *is sup-preserving.*

3.2 One equivalent definition

In this chapter we will look at the definitions introduced in the previous section and reinterpret them in terms of maps between the logic in study and *unsorted equational logic* which was introduced as an example in Section 2.4. In particular we will give an equivalent definition of algebraizable logic. Let us begin with the notion of algebraic semantics, now given from the point of view of maps.

Theorem 3.2.1
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a structural propositional logic and K a class of Σ-algebras. Then K is an algebraic semantics for $\mathcal{L}$ iff there exists a conservative map $\theta : \mathcal{L} \to Eqn^{\Sigma}_{K}$, satisfying the following uniformity condition

> *there exists a set of equations $\Theta = \{\delta_i \approx \lambda_i : i \in I\}$ where $\delta_i, \lambda_i \in T(\Sigma, \{x\})$ such that for all $\varphi \in L_\Sigma$*
> $$\theta(\varphi) = \Theta(x \backslash \varphi).$$

Proof: We will prove each of the implications.

Suppose that $\mathcal{L}$ has an algebraic semantics. Then there exists a class K of Σ-algebras and a set of Σ-equations, $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ where $\delta_j, \lambda_j \in T(\Sigma, \{x\})$ such that, for all $\Gamma \subseteq L_\Sigma$ and $\varphi \in L_\Sigma$ we have that $\Gamma \vdash \varphi$ iff $\Theta[\Gamma] \models^{\Sigma}_{K} \Theta(x \backslash \varphi)$. Consider $\theta : \mathcal{L} \to Eqn^{\Sigma}_{K}$ such that for $\varphi \in L_\Sigma$, $\theta(\varphi) = \Theta(x \backslash \varphi)$. It is obvious that θ is uniform. We now have to prove that θ is a conservative map. Let $\Gamma \subseteq L_\Sigma$ and $\varphi \in L_\Sigma$. Because Θ is a set of defining equations, we have that $\Gamma \vdash \varphi$ iff $\Theta[\Gamma] \models^{\Sigma}_{K} \Theta(x \backslash \varphi)$, this is precisely, $\theta[\Gamma] \models^{\Sigma}_{K} \theta(\varphi)$. Then we conclude that θ is a conservative map.

Suppose now that there exists a uniform conservative map $\theta : \mathcal{L} \to Eqn^{\Sigma}_{K}$ for some class K of Σ-algebras. By the uniformity of θ, there exists a set of equations $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ where $\delta_j, \lambda_j \in T(\Sigma, \{x\})$, such that $\theta(\varphi) = \Theta(x \backslash \varphi)$. We want to prove that K is an algebraic semantics for $\mathcal{L}$ with defining equations Θ. Let $\Gamma \subseteq L_\Sigma$, $\varphi \in L_\Sigma$. Because θ is a conservative map, we have that $\Gamma \vdash \varphi$ iff $\theta[\Gamma] \models^{\Sigma}_{K} \theta(\varphi)$, this is precisely $\Theta[\Gamma] \models^{\Sigma}_{K} \Theta(x \backslash \varphi)$. Then we conclude that K is an algebraic semantics for $\mathcal{L}$ with defining equations Θ.

QED

We stress again the importance of looking at the definition of algebraic semantic as the existence of a uniform map between $\mathcal{L}$ and the unsorted equational logic Eqn^{Σ}_{K} for some class K of Σ-algebras.

We now give an equivalent definition of algebraizable logic in terms of maps between the target logic and Eqn^{Σ}_{K} for some class K of Σ-algebras.

Theorem 3.2.2
A structural propositional logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is algebraizable iff there exists a strong representation (θ, τ) of $\mathcal{L}$ in Eqn^{Σ}_{K}, for some class K of Σ-algebras, such that θ is uniform and τ satisfies the following uniformity condition:

> *there exists a set $E \subseteq L_\Sigma(\{\xi_1, \xi_2\})$ such that for all Σ-equation $\varphi \approx \psi$*
> $$\tau(\varphi \approx \psi) = E(\xi_1 \backslash \varphi,\ \xi_2 \backslash \psi).$$

Proof: We prove each of the implications.

Suppose first that $\mathcal{L}$ is algebraizable. Then there exists a class K of Σ-algebras such that K is an algebraic semantics for $\mathcal{L}$ with defining equations $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ where $\delta_j, \lambda_j \in T(\Sigma, \{x\})$, and there exists a set of equivalence formulas $E \subseteq L_\Sigma(\{\xi_1, \xi_2\})$.

Take $\theta : \mathcal{L} \to Eqn_K^\Sigma$ where $\theta(\varphi) = \Theta(x\backslash\varphi)$ and $\tau : Eqn_K^\Sigma \to \mathcal{L}$ where $\tau(\varphi \approx \psi) = E(\xi_1\backslash\varphi,\ \xi_2\backslash\psi)$. We will show that (θ, τ) is a strong representation of $\mathcal{L}$ in Eqn_K^Σ.

It is obvious that both θ and τ satisfy their respective uniformity condition. Let us now prove that θ and τ are both conservative maps.

Let $\Gamma \cup \{\varphi\} \subseteq L_\Sigma$. Then, because Θ is a set of defining equations, $\Gamma \vdash \varphi$ iff $\Theta[\Gamma] \vDash_K^\Sigma \Theta(x\backslash\varphi)$ and this is precisely $\theta[\Gamma] \vDash_K^\Sigma \theta(\varphi)$, proving that θ is a conservative map.

Let Θ' be any set of Σ-equations and $\varphi \approx \psi$ any Σ-equation. Then, because E is a set of equivalence formulas, we have that $\Theta' \vDash_K^\Sigma \varphi \approx \psi$ iff $E[\Theta'] \vdash E(\xi_1\backslash\varphi,\ \xi_2\backslash\psi)$ and this is precisely $\tau[\Theta'] \vdash \tau(\varphi \approx \psi)$, proving that τ is a conservative map.

Let $\varphi \in L_\Sigma$. We have to prove that $\varphi \dashv\vdash \tau[\theta(\varphi)]$. We know that $\xi \dashv\vdash E[\Theta(x\backslash\xi)]$. Using Structurality we have that $\varphi \dashv\vdash E[\Theta(x\backslash\varphi)]$, but this is precisely $\varphi \dashv\vdash \tau[\theta(\varphi)]$.

Let $\varphi \approx \psi$ be any Σ-equation. We have to prove that $\varphi \approx \psi \ {=}\!\!\mid\!\!\vDash_K^\Sigma \theta[\tau(\varphi \approx \psi)]$. We know that $x_1 \approx x_2 \ {=}\!\!\mid\!\!\vDash_K^\Sigma \Theta[E(\xi_1\backslash x_1,\ \xi_2\backslash x_2)]$. Using Structurality we have that $\varphi \approx \psi \ {=}\!\!\mid\!\!\vDash_K^\Sigma \Theta[E(\xi_1\backslash\varphi, \xi_2\backslash\psi)]$, but this is precisely $\varphi \approx \psi \ {=}\!\!\mid\!\!\vDash_K^\Sigma \theta[\tau(\varphi \approx \psi)]$.

So, (θ, τ) is a strong representation of $\mathcal{L}$ in Eqn_K^Σ.

Suppose now that there exists a strong representation (θ, τ) of $\mathcal{L}$ in Eqn_K^Σ for some class K of Σ-algebras, such that both θ and τ are uniform. Then there exists a set of Σ-equations $\Theta = \{\delta_j \approx \lambda_j : j \in J\}$ where $\delta_j, \lambda_j \in T(\Sigma, \{x\})$ such that, for all $\varphi \in L_\Sigma$, $\theta(\varphi) = \Theta(x\backslash\varphi)$. There exists also a set of formulas $E \subseteq L_\Sigma(\{\xi_1, \xi_2\})$, such that, for all Σ-equation $\varphi \approx \psi$ we have that $\tau(\varphi \approx \psi) = E(\xi_1\backslash\varphi, \xi_2\backslash\psi)$. We aim to prove that K is an algebraic semantics for $\mathcal{L}$ with Θ the set of defining equations and that E is a set of equivalential formulas.

Let $\Gamma \cup \{\varphi\} \subseteq L_\Sigma$. Because θ is a conservative map, we have that $\Gamma \vdash \varphi$ iff $\theta[\Gamma] \vDash_K^\Sigma \theta(\varphi)$, but this is the same that $\Theta[\Gamma] \vDash_K^\Sigma \theta(x\backslash\varphi)$.

Let Θ' be any set of Σ-equations and $\varphi \approx \psi$ any Σ-equation. Because τ is a conservative map, we have that $\Theta' \vDash_K^\Sigma \varphi \approx \psi$ iff $\tau[\Theta'] \vdash \tau(\varphi \approx \psi)$, but this is the same that $E[\Theta'] \vdash E(\xi_1\backslash\varphi, \xi_2\backslash\psi)$.

Let us now prove that $\xi \dashv\vdash E[\Theta(x\backslash\xi)]$. We know that, for all $\varphi \in L_\Sigma$, $\varphi \dashv\vdash \tau[\theta(\varphi)]$, in particular $\xi \dashv\vdash \tau[\theta(\xi)]$, but this is precisely, $\xi \dashv\vdash E[\Theta(x\backslash\xi)]$.

Finally, we will prove that $x_1 \approx x_2 \mathrel{=\!\mid\!\models}^{\Sigma}_{K} \Theta[E(\xi_1\backslash x_1,\ \xi_2\backslash x_2)]$. We know that, for Σ-equation $\varphi \approx \psi$, we have that $\varphi \approx \psi \mathrel{=\!\mid\!\models}^{\Sigma}_{K} \theta[\tau(\varphi \approx \psi)]$. In particular we get that $x_1 \approx x_2 \mathrel{=\!\mid\!\models}^{\Sigma}_{K} \theta[\tau(x_1 \approx x_2)]$, which is the same as $x_1 \approx x_2 \mathrel{=\!\mid\!\models}^{\Sigma}_{K} \Theta[E(\xi_1\backslash x_1,\ \xi_2\backslash x_2)]$.

QED

Chapter 4

Beyond algebraization

4.1 Introduction

In the previous chapter we gave a definition of algebraizable logic in terms of maps between the target logic and unsorted equational logic Eqn_K^Σ for some class K of Σ-algebras. What happens if we replace Eqn_K^Σ by another logic? Which properties of unsorted equational logic are really relevant to the process of algebraization? These are the questions we will explore in this chapter.

4.2 $\mathcal{B}$-able logic

We now consider $\mathcal{B} = \langle L_\mathcal{B}, \vdash \rangle$ a logic and $\mathcal{L} = \langle \Sigma, \vdash \rangle$ a structural propositional logic, that we call the base logic and the object logic, respectively. Underlying $L_\mathcal{B}$ there exists a many-sorted signature $\Sigma' = \langle S, O \rangle$ with distinguished $l, b \in S$ so that

1. $L_\mathcal{B} = gT(\Sigma')_b$;
2. "_" $: L_\Sigma \to T_\mathcal{B} = gT(\Sigma')_l$;
3. "_ $\approx$ _" $: T_\mathcal{B} \times T_\mathcal{B} \to 2^{L_\mathcal{B}}$ uniform in the sense that there exists a set $\Theta \subseteq T(\Sigma', \{x_1, x_2\})_b$ such that "$t_1 \approx t_2$" $= \Theta(x_1 \backslash t_1,\ x_2 \backslash t_2)$;
4. "$t_1 \approx t_2$"$, \Delta(x\backslash t_1) \vdash_\mathcal{B} \Delta(x\backslash t_2)$ para $\Delta \subseteq T(\Sigma', \{x\})_b$
5. for all φ and $\psi \in L_\Sigma$:
    - $\vdash_\mathcal{B}$ "$\varphi \approx \varphi$"
    - "$\varphi \approx \psi$" $\vdash_\mathcal{B}$ "$\psi \approx \varphi$"
    - "$\varphi \approx \psi$", "$\psi \approx \delta$" $\vdash_\mathcal{B}$ "$\varphi \approx \delta$"

where, for $\varphi, \psi \in L_\Sigma$, we abbreviate "“$\varphi$” $\approx$ “ψ”" by “$\varphi \approx \psi$”.

Item 5 says that this “ equality ” induces an equivalence relation between $\mathcal{L}$-formulas.

When we generalize the definition of algebraizable logic by substituting Eqn_K^Σ by another base logic $\mathcal{B}$, we are not interested in any kind of logic. So, we imposed these conditions to the logic $\mathcal{B}$ in order to maintain the “ equational flavor ” of the base logic. Note that clearly Eqn_K^Σ satisfies these conditions.

Definition 4.2.1 (Congruent context)
A *congruent context* is a pair $\langle \delta, \xi \rangle$, where $\delta \in L_\Sigma$ and $\xi \in \Xi$, such that:

$$\text{“}\varphi \approx \psi\text{”} \vdash_{\mathcal{B}} \text{“}\delta(\xi\backslash\varphi) \approx \delta(\xi\backslash\psi)\text{”}.$$

Restricted to the congruent contexts, the relation “$\varphi \approx \psi$” is a *semi-congruence* on $\mathbf{L_\Sigma}$. It is not difficult to see that the set of the semi-congruences on $\mathbf{L_\Sigma}$, denoted by $\text{Semi-Congr}_{\mathbf{L_\Sigma}}$, is a complete partial order, using the same ideas as in the analogue result for congruences over $\mathbf{L_\Sigma}$.

Generalizing the definition of *Eqn*-semantics we get:

Definition 4.2.2 ($\mathcal{B}$-semantics)
A structural propositional logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ has a $\mathcal{B}$*-semantics* if there exists a conservative map θ from $\mathcal{L}$ to $\mathcal{B}$, satisfying the following uniformity condition:

there exists a set $\Theta \subseteq L_{\mathcal{B}}(\{x\})$ such that $\theta(\varphi) = \Theta(x\backslash\text{“}\varphi\text{”})$.

Definition 4.2.3 (Compatibility)
A semi-congruence $\equiv$ on L_Σ is *compatible with a $\mathcal{L}$-theory* Γ provided that if $\varphi \in \Gamma$ and $\langle \varphi, \psi \rangle \in \equiv$ imply $\psi \in \Gamma$.

Definition 4.2.4
Let $Th_{\mathcal{L}}$ be the lattice of the theories of $\mathcal{L}$. Then we can define a function “Ω” so that:

$$\text{“}\Omega\text{”} : Th_{\mathcal{L}} \rightarrow \text{Semi-Congr}_{\mathbf{L_\Sigma}}$$

$$\Gamma \mapsto \text{largest semi-congruence compatible with } \Gamma.$$

We need to prove that “Ω” is well defined, that is, for all $\mathcal{L}$-theory Γ, there exists the largest semi-congruence compatible with Γ.

Proposition 4.2.5
The operator “Ω” can be given as

$$\text{“}\Omega\text{”}(\Gamma) = \{\langle \varphi, \psi \rangle : \textit{ for all congruent context } \langle \delta, \xi \rangle,\ \delta(\xi\backslash\varphi) \in \Gamma \textit{ iff } \delta(\xi\backslash\psi) \in \Gamma\}.$$

Proof:
Let $\Theta(\Gamma) = \{\langle\varphi,\psi\rangle :$ for all congruent context $\langle\delta,\xi\rangle,\ \delta(\xi\backslash\varphi) \in \Gamma$ iff $\delta(\xi\backslash\psi) \in \Gamma\}$. We aim to prove that $\Theta(\Gamma) =$ "Ω"(Γ), that is, $\Theta(\Gamma)$ is the largest semi-congruence compatible with Γ. $\Theta(\Gamma)$ is clearly a semi-congruence and compatible with Γ. We still have to prove that it is the largest one. Let Θ' be a semi-congruence with this property and let $\langle\delta,\xi\rangle$ be any congruent context. Since Θ' is a semi-congruence we have, for every $\langle\varphi,\psi\rangle \in \Theta'$, that

$$\langle\delta(\xi\backslash\varphi), \delta(\xi\backslash\psi)\rangle \in \Theta'.$$

Thus by the compatibility of Θ',

$$\delta(\xi\backslash\varphi) \in \Gamma \text{ iff } \delta(\xi\backslash\psi) \in \Gamma.$$

Hence $\langle\varphi,\psi\rangle \in \Theta(\Gamma)$, and so $\Theta' \subseteq \Theta(\Gamma)$. QED

Definition 4.2.6 (Proto-$\mathcal{B}$-able logic)
A structural propositional logic $\mathcal{L} = \langle\Sigma,\vdash\rangle$ is *proto-B-able* if

$$\langle\varphi,\psi\rangle \in \text{“}\Omega\text{”}(\Gamma) \text{ implies } \Gamma\cup\{\varphi\} \dashv\vdash \Gamma\cup\{\psi\}.$$

Proposition 4.2.7 *$\mathcal{L}$ is proto-$\mathcal{B}$-able if an only if "Ω" is monotone.*

Proof: We will prove each of the implications.

Suppose that $\mathcal{L}$ is proto-B-able.

Consider $\Gamma \subseteq \Delta$. It's enough to prove that "Ω"(Γ) is compatible with Δ. Let $\langle\varphi,\psi\rangle \in$ "Ω"(Γ). Then $\Gamma,\varphi \vdash \psi$ and $\Gamma,\psi \vdash \varphi$. We have then that

if $\varphi \in \Delta$ then $\Gamma\cup\{\varphi\} \subseteq \Delta$ then, since Δ is a theory and $\Gamma,\varphi \vdash \psi$, we have that $\psi \in \Delta$;

and in the same fashion for the case where $\psi \in \Delta$.

Suppose now that Ω is monotone.

Let $\langle\varphi,\psi\rangle \in$ "Ω"(Γ) then, because Ω is monotone, we have that

$\langle\varphi,\psi\rangle \in$ "Ω"$((\Gamma\cup\{\varphi\})^{\vdash})$. We know that, since "$\Omega$"$((\Gamma\cup\{\psi\})^{\vdash})$ is compatible with $(\Gamma\cup\{\psi\})^{\vdash}$, then $\varphi \in (\Gamma\cup\{\varphi\})^{\vdash}$ iff $\psi \in (\Gamma\cup\{\varphi\})^{\vdash}$. Since $\varphi \in (\Gamma\cup\{\varphi\})^{\vdash}$ we also have that $\psi \in (\Gamma\cup\{\varphi\})^{\vdash}$, which is precisely $\Gamma,\varphi \vdash \psi$;

$\langle\varphi,\psi\rangle \in$ "Ω"$((\Gamma\cup\{\psi\})^{\vdash})$. We know that, since "$\Omega$"$((\Gamma\cup\{\psi\})^{\vdash})$ is compatible with $(\Gamma\cup\{\psi\})^{\vdash}$, then $\varphi \in (\Gamma\cup\{\psi\})^{\vdash}$ iff $\psi \in (\Gamma\cup\{\psi\})^{\vdash}$. Since $\psi \in (\Gamma\cup\{\psi\})^{\vdash}$ we also have that $\varphi \in (\Gamma\cup\{\psi\})^{\vdash}$, which is precisely $\Gamma,\psi \vdash \varphi$.

QED

Definition 4.2.8 ($\mathcal{B}$-able logic)
A structural propositional logic $\mathcal{L} = \langle \Sigma, \vdash \rangle$ is *$\mathcal{B}$-able* if there exists a strong representation (θ, τ) of $\mathcal{L}$ in $\mathcal{B}$, so that θ is uniform and τ also satisfies the following uniformity condition:

$$\text{for all } \vartheta \in L_{\mathcal{B}}(\{x_1, \ldots, x_n\}) \text{ there exists a set } \Gamma \subseteq L_{\Sigma}(\xi_1, \ldots, \xi_n) \text{ so that}$$
$$\tau(\vartheta(x_1 \backslash \text{``}\varphi_1\text{''}, \ldots, x_n \backslash \text{``}\varphi_n\text{''})) = \Gamma(\xi_1 \backslash \varphi_1, \ldots, \xi_n \backslash \varphi_n)$$

We will now prove some useful lemmas.

Theorem 4.2.9 *Assuming that $\mathcal{L}$ is $\mathcal{B}$-able with strong representation (θ, τ) then θ^{Th} is sup-preserving and injective.*

Proof: θ^{Th} is sup-preserving by lemma 2.2.4 and is injective by lemma 2.2.8.
QED

In the following we will assume that $\langle \tau(\text{``}\varphi \approx \xi\text{''}), \xi \rangle$ is a congruent context. This is a natural condition and is verified in all examples that we have in mind, in particular in the case of $\mathcal{C}_1$. We believe this result follows immediately from a natural strengthening of the definition of $\mathcal{B}$-able logic, namely the assumption that $\Gamma \subseteq L_{\Sigma}(\xi_1, \ldots, \xi_n)$ is a congruent context in each variable.

Lemma 4.2.10
Assuming that $\mathcal{L}$ is $\mathcal{B}$-able with strong representation (θ, τ) and supposing ξ does not occur in γ_1, we have that $\tau(\text{``}\gamma_1 \approx \xi\text{''})(\xi \backslash \gamma_2) = \tau(\text{``}\gamma_1 \approx \gamma_2\text{''})$.

Theorem 4.2.11
If $\mathcal{L}$ is $\mathcal{B}$-able with (θ, τ) then

$$\text{``}\Omega\text{''}(\Gamma) = \{\langle \gamma_1, \gamma_2 \rangle : \text{``}\gamma_1 \approx \gamma_2\text{''} \subseteq \theta^{Th}(\Gamma)\}.$$

Proof: We prove the two inclusions.

> $\text{``}\Omega\text{''}(\Gamma) \subseteq \{\langle \gamma_1, \gamma_2 \rangle : \text{``}\gamma_1 \approx \gamma_2\text{''} \subseteq \theta^{Th}(\Gamma)\}$.
>
> Let Γ be a $\mathcal{L}$-theory and $\langle \gamma_1, \gamma_2 \rangle \in \text{``}\Omega\text{''}(\Gamma)$. Then for all congruent context $\langle \delta, \xi \rangle$ we have that $\delta(\xi \backslash \gamma_1) \in \Gamma$ iff $\delta(\xi \backslash \gamma_2) \in \Gamma$. Clearly $\text{``}\gamma_1 \approx \gamma_1\text{''} \subseteq \theta^{Th}(\Gamma)$ and, since $\theta^{Th} = \tau^{-1}$, we conclude that $\tau(\text{``}\gamma_1 \approx \gamma_1\text{''}) \subseteq \Gamma$. Using the fact that $\langle \tau(\text{``}\gamma_1 \approx \xi\text{''}), \xi \rangle$ is a congruent context when ξ do not occur in γ_1, we have that $\tau(\text{``}\gamma_1 \approx \xi\text{''})(\xi \backslash \gamma_1) \subseteq \Gamma$ iff $\tau(\text{``}\gamma_1 \approx \xi\text{''})(\xi \backslash \gamma_2) \subseteq \Gamma$. Since we already know that $\tau(\text{``}\gamma_1 \approx \gamma_1\text{''}) \subseteq \Gamma$ then we conclude that $\tau(\text{``}\gamma_1 \approx \gamma_2\text{''}) \subseteq \Gamma$. Using Reflexivity we get that $\Gamma \vdash \tau(\text{``}\gamma_1 \approx \gamma_2\text{''})$. Since $\theta$ is a conservative map and using the fact that $\theta[\tau(\text{``}\gamma_1 \approx \gamma_2\text{''})] \dashv\vdash_{\mathcal{B}} \text{``}\gamma_1 \approx \gamma_2\text{''}$, we conclude that $\text{``}\gamma_1 \approx \gamma_2\text{''} \subseteq \theta^{Th}(\Gamma)$.

$\{\langle\gamma_1, \gamma_2\rangle : \text{“}\gamma_1 \approx \gamma_2\text{”} \subseteq \theta^{Th}(\Gamma)\} \subseteq \text{“}\Omega\text{”}(\Gamma)$.

Suppose Γ is a $\mathcal{L}$-theory and take $\langle\varphi, \psi\rangle \in \{\langle\gamma_1, \gamma_2\rangle : \text{“}\gamma_1 \approx \gamma_2\text{”} \subseteq \theta^{Th}(\Gamma)\}$. Then “$\varphi \approx \psi$” $\subseteq \theta^{Th}(\Gamma)$, that is $\theta[\Gamma] \vdash_{\mathcal{B}}$ “$\varphi \approx \psi$”. Let $\langle\delta, \xi\rangle$ be any congruent context. Then “$\varphi \approx \psi$” $\vdash_{\mathcal{B}}$ “$\delta(\xi\backslash\varphi) \approx \delta(\xi\backslash\psi)$”. Using Cut we conclude that $\theta[\Gamma] \vdash_{\mathcal{B}}$ “$\delta(\xi\backslash\varphi) \approx \delta(\xi\backslash\psi)$”. Now suppose that $\delta(\xi\backslash\varphi) \in \Gamma$. We aim to prove that $\delta(\xi\backslash\psi) \in \Gamma$. Since Γ is a $\mathcal{L}$-theory we have that $\Gamma \vdash \delta(\xi\backslash\varphi)$. We know that there exists a $\Theta \subseteq L_{\mathcal{B}}(\{x\})$ such that, for all $\gamma \in L_\Sigma$, we have that $\theta(\gamma) = \Theta(x\backslash\gamma)$. Then by conservativeness of θ we conclude that $\theta[\Gamma] \vdash_{\mathcal{B}} \theta(\delta(\xi\backslash\varphi)) = \Theta(x\backslash\delta(\xi\backslash\varphi))$ and using property 4 of $\mathcal{B}$ we have that $\theta[\Gamma] \vdash_{\mathcal{B}} \Theta(x\backslash\delta(\xi\backslash\psi)) = \theta(\delta(\xi\backslash\psi))$. Again by the conservativeness of θ we conclude that $\Gamma \vdash \delta(\xi\backslash\psi)$, and since Γ is a $\mathcal{L}$-theory we get that $\delta(\xi\backslash\psi) \in \Gamma$.

In the same way we can prove that if $\delta(\xi\backslash\psi) \in \Gamma$ then $\delta(\xi\backslash\varphi) \in \Gamma$. So we conclude that for all congruent contexts $\langle\delta, \xi\rangle$, we have that $\delta(\xi\backslash\varphi) \in \Gamma$ iff $\delta(\xi\backslash\psi) \in \Gamma$, that is, $\langle\varphi, \psi\rangle \in$ “Ω”(Γ).

QED

Chapter 5

Example

One very interesting example where the theory of algebraization of logics does not apply are non-truth-functional logics. In particular we have the paraconsistent logics of daCosta [dC63, dC74], as it was proved in [Mor80] and in [LMS91]. The main motivation for this work was precisely the extension of the theory in order to handle this kind of logic. In this chapter, we will show that the paraconsistent logic $\mathcal{C}_1$ is $\mathcal{B}$-able, where $\mathcal{B}$ is a two-sorted equational logic. This idea of using two-sorted algebras of formulas and truth-values and including a valuation map as an operator between the two sorts was first used in [CCC+03].

5.1 The example of $\mathcal{C}_1$

Consider the paraconsistent logic $\mathcal{C}_1$ given as an example in Chapter 2.4.1. In this example we will use the two-sorted equational logic $Eqn_K^{\Sigma^{\phi,\tau}}$ where we take as K the family of algebras such that the truth-values algebra is a boolean algebra. By simplicity of notation we will denote this logic by $Eqn_{Bool}^{\phi,\tau}$. The two-sorted signature $\Sigma^{\phi,\tau} = \langle S, O\rangle$ of $Eqn_{Bool}^{\phi,\tau}$, induced by the propositional signature Σ of $\mathcal{C}_1$ and the usual signature of Boolean algebras, is as follows:

- $S = \{\phi, \tau\}$
- $O_{\epsilon\phi} = \Sigma_0 \cup \Xi$;
- $O_{\phi^k\phi} = \Sigma_k$ for $k > 0$;
- $O_{\phi\tau} = \{v\}$;
- $O_{\epsilon\tau} = \{\top, \bot\}$;
- $O_{\tau\tau} = \{-\}$;
- $O_{\tau\tau\tau} = \{\sqcap, \sqcup, \Rightarrow\}$;

- $O_w = \varnothing$ otherwise;

As we remarked in Chapter 2.4, since the family of boolean algebras is a variety, we will work in this example with the sintatical consequence relation $\vdash_{Eqn_2}$ of the usual two-sorted equational consequence relation plus the addition of the following set Ax of axioms that specifies the family of boolean algebras and, since we want the valuation to be non-truth-functional, we also have valuation axioms.

- Truth values axioms - specification of the class of all Boolean algebras:

  ⋆ $x_1 \sqcap x_2 \approx x_2 \sqcap x_1$

  ⋆ $x_1 \sqcup x_2 \approx x_2 \sqcup x_1$

  ⋆ $x_1 \sqcap (x_2 \sqcap x_3) \approx (x_1 \sqcap x_2) \sqcap x_3$

  ⋆ $x_1 \sqcup (x_2 \sqcup x_3) \approx (x_1 \sqcup x_2) \sqcup x_3$

  ⋆ $x \sqcap x \approx x$

  ⋆ $x \sqcup x \approx x$

  ⋆ $(x_1 \sqcap x_2) \sqcup x_2 \approx x_2$

  ⋆ $(x_1 \sqcup x_2) \sqcap x_1 \approx x_1$

  ⋆ $x_1 \sqcap (x_2 \sqcup x_3) \approx (x_1 \sqcap x_2) \sqcup (x_1 \sqcap x_3)$

  ⋆ $x_1 \sqcup (x_2 \sqcap x_3) \approx (x_1 \sqcup x_2) \sqcap (x_1 \sqcup x_3)$

  ⋆ $x \sqcup \top \approx \top$

  ⋆ $\bot \sqcap x \approx \bot$

  ⋆ $-(-x) \approx x$

  ⋆ $x \sqcap (-x) \approx \bot$

  ⋆ $x \sqcup (-x) \approx \top$

  ⋆ $-(x_1 \sqcap x_2) \approx (-x_1) \sqcup (-x_2)$

  ⋆ $-(x_1 \sqcup x_2) \approx (-x_1) \sqcap (-x_2)$

  ⋆ $x_1 \Rightarrow x_2 \approx (-x_1) \sqcup x_2$

The problem with the the logic $\mathcal{C}_1$ in terms of the classical theory is that the negation connective is non-truth-functional. So the valuation $v \in O_{\phi\tau}$ cannot be defined as an homomorphism. Indeed we have to impose with axioms the properties we which our valuation should satisfy.

- Valuation axioms:

  ⋆ $v(\mathbf{t}) \approx \top$

- $\star$ $v(\mathbf{f}) \approx \bot$
- $\star$ $v(y_1 \wedge y_2) \approx v(y_1) \sqcap v(y_2)$
- $\star$ $v(y_1 \vee y_2) \approx v(y_1) \sqcup v(y_2)$
- $\star$ $v(y_1 \supset y_2) \approx v(y_1) \Rightarrow v(y_2)$
- $\star$ $-v(y_1) \leqslant v(\neg y_1)$
- $\star$ $v(\neg\neg y_1) \leqslant v(y_1)$
- $\star$ $(v(y_1^\circ) \sqcap (v(y_1) \sqcap v(\neg y_1))) \approx \bot$
- $\star$ $(v(y_1^\circ) \sqcap v(y_2^\circ)) \leqslant v((y_1 \wedge y_2)^\circ)$
- $\star$ $(v(y_1^\circ) \sqcap v(y_2^\circ)) \leqslant v((y_1 \vee y_2)^\circ)$
- $\star$ $(v(y_1^\circ) \sqcap v(y_2^\circ)) \leqslant v((y_1 \supset y_2)^\circ)$

Where we define $\leqslant$ as: $t_1 \leqslant t_2$ iff $t_1 \sqcap t_2 \approx t_1$. It is important to note that $v(\neg y) \approx -v(y)$ does not follow.

First of all, note that, using "φ" $= v(\varphi)$ and "$\varphi \approx \psi$" $= \{v(\varphi) \approx v(\psi)\}$, the two-sorted equational logic that we have introduce in Chapter 2 satisfies the requirements that we have imposed, in Chapter 4, to the base logic. Now we will show that the logic $\mathcal{C}_1$ is $Eqn^{\phi,\tau}_{Bool}$-able. So, all we have to do is define a strong representation of $\mathcal{C}_1$ in $Eqn^{\phi,\tau}_{Bool}$.

Proposition 5.1.1
$\mathcal{C}_1$ is $Eqn^{\phi,\tau}_{Bool}$-able.

Proof: Consider the pair (θ, τ) with

- $\theta : \mathcal{C}_1 \to Eqn^{\phi,\tau}_{Bool}$ such that $\varphi \stackrel{\theta}{\longmapsto} \{v(\varphi) \approx \top\}$; and
- $\tau : Eqn^{\phi,\tau}_{Bool} \to \mathcal{C}_1$ such that $t_1 \approx t_2 \stackrel{\tau}{\longmapsto} \{t_1^* \supset t_2^*,\ t_2^* \supset t_1^*\}$

where t^* is defined inductively in the following way:

. $v(\varphi)^* = \varphi$;

. $\top^* = \mathbf{t}$;

. $\bot^* = \mathbf{f}$;

. $(-t)^* = t^* \supset \mathbf{f}$;

. $(t_1 \sqcap t_2)^* = t_1^* \wedge t_2^*$;

. $(t_1 \sqcup t_2)^* = t_1^* \vee t_2^*$;

. $(t_1 \Rightarrow t_2)^* = t_1^* \supset t_2^*$

If Θ is a set of equations, Θ^* denotes the set $\{eq^* : eq \in \Theta\}$.

It is trivial to see that both θ and τ satisfy their correspondent uniformity condition. Note also that, like we remarked after the definition of $\mathcal{B}$-able logic, it is easy to see that, in this particular example, $\langle\tau(\text{“}\varphi \approx \xi\text{”}), \xi\rangle$ is a congruent context, since $\tau(\text{“}\varphi \approx \xi\text{”}) = \tau(v(\varphi) \approx v(\xi)) = (\varphi \equiv \xi)$.

We still have to prove that the pair (θ, τ) is a strong representation between $\mathcal{C}_1$ and $Eqn_{Bool}^{\phi,\tau}$. Like we remarked after the definition of strong representation, it is enough to prove that θ is a conservative map, and also that $t_1 \approx t_2 \dashv\vdash_{Eqn_2} \theta[\tau(t_1 \approx t_2)]$. First, let us prove that θ is a conservative map.

θ is a conservative map:

We aim to prove that $\Gamma \vdash_{\mathcal{C}_1} \varphi$ iff $\theta[\Gamma] \vdash_{Eqn_2} \theta(\varphi)$. Let us first show that if $\Gamma \vdash_{\mathcal{C}_1} \varphi$ then $\theta[\Gamma] \vdash_{Eqn_2} \theta(\varphi)$. Suppose $\Gamma \vdash_{\mathcal{C}_1} \varphi$. We want to see that $\theta(\Gamma) \vdash_{Eqn_2} \theta(\varphi)$, that is, $\{v(\gamma) \approx \top : \gamma \in \Gamma\} \vdash_{Eqn_2} v(\varphi) \approx \top$. We will prove this by induction in the length of the derivation of φ from Γ in $\mathcal{C}_1$. For this, we will prove first that for each axiom φ of $\mathcal{C}_1$ we have that $\vdash_{Eqn_2} v(\varphi) \approx \top$.

We will not prove this in detail for all the axioms, since some of the proofs are analogue and straightforward. The first two axioms of $\mathcal{C}_1$ are axioms of CPL then, by the correction of CPL with respect to Boolean algebras, and the adequacy of Eqn_2, we get the result. We will only give the idea how the rigorous derivations can be done. We will do this so that the full detailed derivation can be easily obtained.

Let us prove that the third axiom $((\xi_1 \wedge \xi_2) \supset \xi_1)$ is correct.

We want to prove that $\vdash_{Eqn_2} v((y_1 \wedge y_2) \supset y_1) \approx \top$.

$$\begin{aligned}
v((y_1 \wedge y_2) \supset y_1) &\approx v(y_1 \wedge y_2) \Rightarrow v(y_1)\\
&\approx v(y_1) \sqcap v(y_2)) \Rightarrow v(y_1)\\
&\approx (-(v(y_1) \sqcap v(y_2))) \sqcup v(y_1)\\
&\approx ((-v(y_1)) \sqcup (-v(y_2))) \sqcup v(y_1)\\
&\approx ((-v(y_2)) \sqcup (-v(y_1))) \sqcup v(y_1)\\
&\approx (-v(y_2)) \sqcup ((-v(y_1)) \sqcup v(y_1))\\
&\approx -v(y_2) \sqcup \top\\
&\approx \top
\end{aligned}$$

The axioms 4,5,6,7,8 are handled similarly.

Let us now prove that $\vdash_{Eqn_2} v(\neg\neg y_1 \supset y_1) \approx \top$

$$\begin{aligned} v(\neg\neg y_1 \supset y_1) &\approx v(\neg\neg y_1) \Rightarrow v(y_1) \\ &\approx (-v(\neg\neg y_1)) \sqcup v(y_1) \\ &\approx (-(v(\neg\neg y_1) \sqcap v(y_1))) \sqcup v(y_1) \\ &\approx ((-v(\neg\neg y_1)) \sqcup (-v(y_1))) \sqcup v(y_1) \\ &\approx (-v(\neg\neg y_1)) \sqcup ((-v(y_1)) \sqcup v(y_1)) \\ &\approx (-v(\neg\neg y_1)) \sqcup \top \\ &\approx \top \end{aligned}$$

Let us now prove that $\vdash_{Eqn_2} v((y_1^\circ \wedge y_2^\circ) \supset (y_1 \wedge y_2)^\circ) \approx \top$.
We get that

$$\begin{aligned} v((y_1^\circ \wedge y_2^\circ) \supset (y_1 \wedge y_2)^\circ) &\approx (v(y_1^\circ) \sqcap v(y_2^\circ)) \Rightarrow v((y_1 \wedge y_2)^\circ) \\ &\approx ((v(y_1^\circ) \sqcap v(y_2^\circ)) \sqcap (v((y_1 \wedge y_2)^\circ))) \Rightarrow v((y_1 \wedge y_2)^\circ) \\ &\approx -((v(y_1^\circ) \sqcap v(y_2^\circ)) \sqcap (v((y_1 \wedge y_2)^\circ))) \sqcup v((y_1 \wedge y_2)^\circ) \\ &\approx (-(v(y_1^\circ) \sqcap v(y_2^\circ)) \sqcup -(v((y_1 \wedge y_2)^\circ))) \sqcup v((y_1 \wedge y_2)^\circ) \\ &\approx -(v(y_1^\circ) \sqcap v(y_2^\circ)) \sqcup (-(v((y_1 \wedge y_2)^\circ)) \sqcup v((y_1 \wedge y_2)^\circ)) \\ &\approx -(v(y_1^\circ) \sqcap v(y_2^\circ)) \sqcup (-(v((y_1 \wedge y_2)^\circ)) \sqcup v((y_1 \wedge y_2)^\circ)) \\ &\approx -(v(y_1^\circ) \sqcap v(y_2^\circ)) \sqcup \top \\ &\approx \top \end{aligned}$$

The remaining axioms are treated in a very similar way.

After treating the axioms, we will now prove that the rule $\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle$ preserves conservativeness, that is, $\{v(y_1) \approx \top, v(y_1 \supset y_2) \approx \top\} \vdash_{Eqn_2} v(y_2) \approx \top$. From $v(y_1 \supset y_2) \approx \top$, using the valuation axioms, we get $v(y_1) \Rightarrow v(y_2) \approx \top$. Using the last boolean axiom and transitivity we get $(-v(y_1)) \sqcup v(y_2) \approx \top$. Again in an informal way consider the sequence:

$$\begin{aligned} v(y_2) &\approx \bot \sqcup v(y_2) \\ &\approx -(\top) \sqcup v(y_2) \\ &\approx -(v(y_1)) \sqcup v(y_2) \\ &\approx \top \end{aligned}$$

So we conclude that $\{v(y_1) \approx \top, v(y_1 \supset y_2) \approx \top\} \vdash_{Eqn_2} v(y_2) \approx \top$.

Recall that our intention is to prove that if $\Gamma \vdash_{\mathcal{C}_1} \varphi$ then $\theta(\Gamma) \vdash_{Eqn_2} \theta(\varphi)$. As we said, we will prove this by induction on the length n of the derivation of φ from Γ in $\mathcal{C}_1$.

Suppose first that $n = 1$. Then we have two possibilities:

- $\varphi \in \Gamma$.
  The result follows from the fact that $v(\varphi) \approx \top \in \theta(\Gamma)$.
- φ is an axiom.
  We have proved that $\vdash_{Eqn_2} \theta(\varphi)$ for all axiom φ of $\mathcal{C}_1$. Then using

monotonicity we have the result.

Suppose now, by induction hypothesis, that, for all ψ such that there exists a derivation of ψ from Γ with length less than n, we have that $\theta(\Gamma) \vdash_{Eqn_2} \theta(\psi)$. Let´s look again at the derivation of φ from Γ. We have three cases:

- $\varphi \in \Gamma$.
  Equal to the base
- φ is an axiom.
  Equal to the base
- φ is obtained by the rule $\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle$ from ψ and $\psi \supset \varphi$, both with derivations from Γ with length less than n. By induction hypothesis, we have that $\theta[\Gamma] \vdash_{Eqn_2} v(\psi) \approx \top$ and $\theta[\Gamma] \vdash_{Eqn_2} v(\psi \supset \varphi) \approx \top$. Since we proved that the rule $\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle$ preserves conservativeness we have that $\theta[\Gamma] \vdash_{Eqn_2} v(\varphi) \approx \top$.

We still have to prove the reverse implication, that is, if $\theta[\Gamma] \vdash_{Eqn_2} \theta(\varphi)$ then $\Gamma \vdash_{\mathcal{C}_1} \varphi$. We will prove this by contraposition, proving that if $\Gamma \nvdash_{\mathcal{C}_1} \varphi$ then $\theta[\Gamma] \nvdash_{Eqn_2} \theta(\varphi)$. Recall that the logic $\mathcal{C}_1$ is complete for the semantic of bivaluations. Assume that $\Gamma \nvdash_{\mathcal{C}_1} \varphi$. Then there exists a bivaluation v such that v is a model of Γ but v does not satisfy φ. Now just observe that bivaluations are particular cases of our models, with just the 2-valued Boolean algebra. Then the 2-valued Boolean algebra and the valuation v show that $\theta[\Gamma] \nvDash^{\Sigma^{\phi,\tau}}_{Bool} \theta(\varphi)$. By the soundness of Eqn_2 we have that $\theta[\Gamma] \nvdash_{Eqn_2} \theta(\varphi)$. After proving that θ is a conservative map, we will prove that $t_1 \approx t_2 \dashv\vdash_{Eqn_2} \theta[\tau(t_1 \approx t_2)]$. For this let us present two useful lemmas (see [CCC+03] for details).

Lemma 5.1.2 *If* $t \in T(\Sigma_2)_\tau$, *then* $\vdash_{Eqn_2} v(t^*) \approx t$.

Lemma 5.1.3 *If* $t_1, t_2 \in T(\Sigma_2)_\tau$, $\mathbf{A}$ *is a* Σ_2*-algebra that satisfies the boolean and the valuation axioms and* ρ *is a assignment over* $\mathbf{A}$ *then*

$$[\![t_1]\!]^\rho_{\mathbf{A}} = [\![t_2]\!]^\rho_{\mathbf{A}} \quad \textit{iff} \quad \underline{v}_{\mathbf{A}}([\![t_1^* \equiv t_2^*]\!]^\rho_{\mathbf{A}}) = \underline{\top}_{\mathbf{A}}$$

The last lemma says precisely that $t_1 \approx t_2 \vDash^{\Sigma^{\phi,\tau}}_{Bool} (t_1^* \equiv t_2^*) \approx \top$ and also that $(t_1^* \equiv t_2^*) \approx \top \vDash^{\Sigma^{\phi,\tau}}_{Bool} t_1 \approx t_2$. This is same as $t_1 \approx t_2 \Dashv\vDash^{\Sigma^{\phi,\tau}}_{Bool} (t_1^* \equiv t_2^*) \approx \top$. In terms of the maps θ and τ, this is exactly $t_1 \approx t_2 \Dashv\vDash^{\Sigma^{\phi,\tau}}_{Bool} \theta[\tau(t_1 \approx t_2)]$. By the adequacy of Eqn_2 we get that $t_1 \approx t_2 \dashv\vdash_{Eqn_2} \theta[\tau(t_1 \approx t_2)]$.

QED

Chapter 6

Conclusions

In Chapter 2 we introduced the notions of logic, maps of logics and gave some examples of logics. In Chapter 3 we gave an alternative characterization of some notions of the existing theory of algebraization of logics, in terms of maps between the target logic and unsorted equational logic.

Our main contribution was the generalization of the theory, given in Chapter 4. We generalized the notions we characterized in Chapter 3, by abstracting away from unsorted equational logic the relevant properties for the algebraization process, and then replacing it by another base logic with these properties.

In Chapter 5 we gave a concrete example of the application of our theory to the paraconsistent logic $\mathcal{C}_1$ of daCosta.

It remained to prove that $\langle \tau(\text{“}\varphi \approx \xi\text{”}), \xi \rangle$ is a congruent context. This is a natural condition and we believe that, with a natural strengthening of the definition of $\mathcal{B}$-able logic, this condition will immediately follow.

Our focus was not on the characterization of all the notions of the algebraization hierarchy but on the characterization of the main notion of the theory, that of algebraizable logic. Nevertheless, we hope that most of these notions could be characterized and therefore generalized, in particular the notion of equivalential logic whose generalization we believe to be the natural converse of the notion of $\mathcal{B}$-semantics.

We also want to know if every proto-$\mathcal{B}$-able logic is $\mathcal{B}$-able, since it is well known that every protoalgebraizable logic is algebraizable, being clear that our effort is also on generalizing the main results of the existing theory.

In the future we intend to study the semantic consequences of this generalization, namely using logical matrices. In particular in the example of $\mathcal{C}_1$ we intend to give an “ algebraic flavor ” to its non-truth-functional semantic of bivaluations, and further recover the so-called “ daCosta algebras ” now using an inequational base. We would also like to study what could be obtain by substituting unsorted equational logic by other interesting base logics, namely multisorted equational logic, inequational logics based on ordered algebras, and logics based on partial algebras and on non-deterministic algebras.

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