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Fusion of General Modal Logics Labelled with Truth Values

Karina Girardi Roggia

Supervisor:	**Doctor Maria Cristina de Sales Viana Serôdio Sernadas**
Co-Supervisor:	**Doctor João Filipe Quintas dos Santos Rasga**

Thesis approved in public session to obtain the PhD Degree in Mathematics

Jury final classification: Pass

Jury

Chairperson: Chairman of the IST Scientific Board

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Título Fusão de Lógicas Modais com Semântica Generalizada Etiquetadas com Valores de Verdade

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Resumo

Fusão é uma forma de combinação de lógicas bem conhecida e amplamente estudada principalmente no contexto de lógicas modais normais sendo que a investigação para lógicas modais não-normais ainda é incipiente. Um dos objetivos deste trabalho é apresentar a fusão de lógicas modais não-normais preservando correção e completude. Para essa finalidade são apresentados sistemas lógicos etiquetados com valores de verdade. O cálculo para estes sistemas é baseado em um cálculo de sequentes etiquetados. Para lógicas que podem ser expressas por estruturas do tipo Kripke, por exemplo, isto significa que as fórmulas podem ser etiquetadas não só por um único mundo, mas na verdade por um conjunto de mundos. A semântica é dada por álgebras bisortidas que permitem capturar a semântica generalizada de lógicas modais. Tanto lógicas modais normais quanto lógicas modais não-normais são então apresentadas neste ambiente. Morfismos entre estes sistemas de lógicas são definidos com o intuito de utilizar uma abordagem categórica para definir a fusão como uma soma amalgamada de dois monomorfismos. A combinação prevista é obtida e provas da preservação da correção e da completude são dadas. Alguns exemplos de sistemas combinados são mostrados, incluindo fusões de duas lógicas modais onde pelo menos uma é não-normal.

Palavras-chave: Lógica modal (normal e não-normal), fusão, semântica generalizada, valores de verdade, sequentes etiquetados, dedução etiquetada, abordagem categorial, correção, completude, resultados de preservação.

Title: Fusion of General Modal Logics Labelled with Truth Values

Abstract:

Fusion is a well know and widely studied form of combining logics. However it is investigated mainly for normal modal logics while the research for non-normal modal logics is still incipient. One of the aims of this work is to introduce fusion of non-normal modal logics preserving soundness and completeness. For this purpose, logic systems labelled with truth values are adopted. The calculus is based in a labelled sequent calculus. For instance, in what concerns logics that can be expressed by Kripke-like structures this means that formulas can be labelled not only by a single world, but actually by a set of worlds. The semantics is given by bisorted algebras, that can capture the general semantics of modal logics. Both normal and non-normal modal logics are then presented in this framework. Morphisms between these logic systems are defined in order to use a categorical approach to define fusion as a pushout of two monomorphisms. The envisaged combination is so obtained and preservation of soundness and completeness is then proved. Some examples of combined logic systems are given, including fusions of two modal logics where at least one is non-normal.

Keywords: (Normal and non-normal) modal logic, fusion, general semantics, truth values, labelled sequents, labelled deduction, categorical approach, soundness, completeness, preservation results.

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I have always thought that this would be the easy part to do when writing the thesis. I was wrong. There are so many people that I would like to say thanks that if I mentioned each one I should be forgetting someone, but I must name some for all that they help me.

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Contents

Chapter 1

Introduction

Fusion is a well know and widely studied form of combining logics [KW91, FS96, KW97, Wol98, Kur06]. However it is investigated mainly for normal modal logics while the research for non-normal modal logics is still incipient [FF05, FF02]. One of the aims of this work is to introduce fusion of non-normal modal logics while preserving soundness and completeness. For this purpose, it is adopted the logic systems labelled with truth values introduced in [MSSV04].

1.1 Background

The three main building subjects of this thesis are modal logics, labelled systems and combination of logics, in particular fusion.

1.1.1 Modal logics

Modal logics started with Lewis defining systems with strict implication instead of material implication [Lew18]. Based in this work [LL32] defines the systems $\mathsf{S1}$–$\mathsf{S5}$, where $\mathsf{S4}$ and $\mathsf{S5}$ are well known normal modal systems. These systems were constructed independently of the classical propositional logic. In fact, it was Gödel that had the idea to build a modal logic system as an extension of the classical propositional calculus, when constructing a system similar to $\mathsf{S4}$ in [Göd33]. Lemmon used this approach to construct Lewis' systems in [Lem57].

Despite that the original motivation of modal systems was to deal with strict implication, for many years modal logic was seen as "the logic of necessity and possibility". Only after the introduction of Kripke semantics they were seen also as "simple yet expressive languages for talking about relational structures" [BdRV01]. A great multitude of modal systems, with various kinds of modal operators, each of them with different meanings or properties, have then been constructed in order to provide effective formalisms for talking about time, space, knowledge, beliefs, actions, obligations, among others.

The interpretation of modal logic simply as "the logic of necessity" restricts not only its applications but also the range of modal systems. In fact, modal logic can also be seen as "the study of *modalities* – logical operations that qualify assertions about the truth of statements" [Gol93]. In this sense, the necessitation rule (from φ infers $\Box\varphi$) can be relaxed. This brings a bunch of modal systems known as non-normal, whose semantics was defined only in the 1970's by Montague [Mon70] and Scott [Sco70]. The Scott-Montague semantics, or also known as *neighbourhood semantics*, uses the notion of set of worlds as Kripke semantics does, but instead of a relation between worlds, frames are composed by a function N from the set of worlds W to the powerset of the powerset of W ($N(w)$ is called the set of neighbours of the world $w \in W$). The satisfaction of a modality $\Box\varphi$ in a certain world w is then given iff the denotation of the formula φ belongs to the neighbours of w.

The use of these "weaker" modal logics can avoid some undesirable properties brought by the normality. In epistemic logics, for instance, normal systems have the problem of logical omniscience [Hoc72]: due to the necessity rule, taking the modality as a knowledge operator, all theorems must be known (in practice, this is not true). In [vBP11] a semantics of evidence is proposed based on neighbourhood models. Similar problems are found either in deontic systems where, for instance, Ross paradox ($\Box\varphi \supset \Box(\varphi \vee \psi)$) should be avoided [CJ02], or when using the modality as a likelihood operator based on some fixed probability (where $\Box\varphi \wedge \Box\psi \supset \Box(\varphi \wedge \psi)$ can be false) [Pac07].

In order to have contact with modal logics the reader can consult the books [Che80, BdRV01, Kra99] dedicated to this subject.

1.1.2 Labelled systems

While the extension of a Hilbert calculus with new axioms is not very complex, the use of this kind of deduction systems is not simple in what concerns theorem proving. In fact, the calculus of a logic system, if designed to be easy for theorem proving, should be presented by as natural deduction, tableaux, or by a sequent system. However this kind of calculus are particularly suitable for classical propositional logic (or for first-order calculus) but they are in general neither uniform nor modular for non-classical logics. In many cases a tailored calculus has to be provided for each modal logic, for instance, in the use of simple sequent calculus [Wan02]. That is, the definition cannot be given using extensions as in the case of Hilbert calculi.

A solution to avoid this problem was to incorporate, in some sense, semantical information at the deduction level. Labelled deduction systems [Gab96b, Vig00, BDG$^+$00, RSSV02b] are based in this idea. In these systems the deduction calculus manipulates pairs *label* : $formula$ instead of reasoning only about formulas. Herein, labels can be worlds, truth values, resources, names of agents, nodes of a net or any piece of structured information that facilitates the derivation process. This idea traces back to the work of Prior [Pri67] and, for the case of modal labelled deduction, to the work of Fitting [Fit83]. The latter work covers deductive systems for most common modal logics and also for a number of regular, non-normal modal logics and some quasi-regular logics as S2 and S3.

Labelled deduction was used to define natural deduction calculus for intuitionistic logic [BMV98b], relevance logics [BMV98b, RSSV02b], substructural logics [BFR99] (this class of logics is defined using labelled tableaux systems in [DG94]), many-valued logics [RSSV02b] (labelled sequent calculus for many-valued logics is presented in [BFSZ98]) and first-order logics [Ras03]. For modal logics the list has several guises: tableaux [BR05], sequent systems [GR00, MSSV04], natural deduction [Rus96, BMV97, BMV98b, RSSV02b, SVRS03, dQG99]. Note that in [RSSV02b, SVRS03, MSSV04] labels are truth values instead of simple worlds. Observe that section 1.6 of [dQG99] mentions the possibility to use labelled systems to define a calculus for non-normal modal logics. Quantified modal logic by labelled natural deduction was proposed in [BMV98a] while [ABGR96, GM02] use tableaux systems. Intuitionistic modal logics was also defined with labelled natural deduction in [Sym94]. With such an extensive list of logics using the labelled deduction approach, the problem of combining them arose naturally, as is analysed in the following subsection.

In order to have contact with labelled systems the reader can consult the books [Gab96b, Vig00, BDG$^+$00] dedicated to this subject.

1.1.3 Combination of logics

Reuse. This word is a slogan in many areas nowadays. *Reuse to preserve the environment*, in ecology. *Reuse to save money*, say economists. *Reuse to don't rewrite code for the same functionality*, from software engineers. So in logics: reuse. This is one of the reasons for combination of logics: get new logics from well known ones, while preserving properties, in order to have more expressive and complex systems and to simplify problems. This is a subject concerning not only mathematical logic but also philosophical and applied logics as well. So preservation of properties from the components is crucial and in fact is the main goal when investigating combination of logics. In fact it is where most of the effort of the research in this field is concentrated.

The concept of get new logics from existing ones goes back to Ramón Lull in the XIII century and to Gottfried W. Leibniz in the XVII century with the idea of building schemes where different logics could cooperate instead of competing. But it was only at the end of the last century that rigorous techniques for combination were developed and studied.

Combination started in modal logics, with the work of Fitting [Fit69] that provided the first examples of fusion of modal logics. However it was Thomason in 1984 that introduced the term and of the technique in [Tho84]. In fusion of normal modal logics there is no interaction between the modalities. On the other hand, in the product of modal logics, introduced independently by [Seg73] and [She78], modalities have interactions like commutativity ($\Diamond\blacklozenge\varphi \supset \blacklozenge\Diamond\varphi$) and the Church-Rosser property ($\Diamond\blacksquare\varphi \supset \blacksquare\Diamond\varphi$) in these logics.

Dov Gabbay in the 1990's introduced the mechanism of fibring of (normal) modal logics [Gab96a, Gab96c] from which fusion is a particular case, and which permits some bridge principles between the logics to be combined. A more general presentation of fibring, the algebraic fibring, was developed in [SSC99] to allow combinations of a wider class of logics using a categorical approach. Categorically, unconstrained fibring (when the signatures of the component logics do not share connectives at all) is defined as a coproduct in the category of logic systems, and constrained fibring is defined as a cocartesian lifting. Research on this subject included preservation of properties [ZSS01, SSZ02, CRS08a, CSS11], the collapsing problem [SRC02] and extending the mechanism to several kinds of logics [CCC$^+$03, RSSV02a, SSCM00, CSS03, CFSS08, SSRC09] just to name a few.

Other forms of combination of logics related in some sense with fusion are temporalization [FG92], parametrization [SSC97] and importing [RSSnt]. The main distinction of these forms of combination in respect to fibring is that their language are intrinsically asymmetric and the rules of the temporalized/parametrizated/imported logic can only be applied to its own formulas.

In order to study combination of logics with a wider scope, one can consult the book [CCG$^+$08] which is very complete technically and contains several examples of applications, or the article [CC11] which is a survey emphasizing the philosophical aspects of combining logics.

1.2 Aims

The main aims of this thesis are:

- to extend the Logic Systems Labelled with Truth Values (LSLTV) approach to a wider class of logics besides normal modal logics;
- to define the fusion of LSLTV;
- to study the preservation of properties of the defined fusion, namely soundness and completeness;
- to combine non-normal modal logics.

1.3 Overview

This thesis is organized in five more chapters in a sequence to introduce definitions and to show results in one chapter and presenting examples in the chapter thereafter.

Chapter 2 is dedicated to the definition of Logic Systems Labelled with Truth Values (LSLTV) and to prove some of their properties. LSLTV are systems where the deduction part is presented by a sequent calculus including assertions about labelled formulas and terms. As in [MSSV04] assertions about terms include symbols to describe components of the intended semantics. The semantics consists of bisorted algebras: one sort for formulas and the other for terms. Soundness and completeness results are proved at the end of the chapter.

In Chapter 3 the framework of LSLTV is used to present modal logic systems: normal and non-normal. Each section in this chapter follows the order established in the previous chapter: the language comes first, followed by the deduction systems and the semantical counterpart, and the section finishes with examples of logic systems. Most of the results of Section 3.1, about normal modal logics, are from [MSSV04] while Section 3.2 is new and concentrates on non-normal modal logics.

Chapter 4 is about fusion of LSLTV. Since fusion is a kind of combination defined in the literature for (normal) modal logics, the chapter starts by presenting the combination for this kind of logic systems labelled with truth values, by defining fusion algebraically in Section 4.1. The section afterwards presents a categorical approach for the definition of fusion in order to extend the kind of systems allowed to combine. For that purpose first it is presented the notion of LSLTV morphism and showed some of its properties. The second part of Section 4.2 gives the definition of categorical fusion for a wider class of systems in comparison with the original presentation. This presentation of fusion not only generalizes the classes of systems that can be combined but also permits sharing of different sets of connectives (and operators, since the framework is LSLTV), so, for instance, a modality can be shared in a fusion. Section 4.3 establishes the conditions to the preservation of soundness and completeness in this combination.

Chapter 5 concentrates on applications of fusion by presenting different combined systems. In Section 5.1 not only fusion of normal modal logic systems is given (with one example of sharing of the modality), but also the fusion of two non-normal modal logic systems, and the fusion of a normal modal logic system with a non-normal one are given. Section 5.2 analyses soundness and completeness of some of these examples and of a particular system obtained by fusion of two non-normal modal logic systems whose modalities have strong interactions.

The last chapter compiles the main contributions of this thesis with some final remarks and pointing several directions of research from the results here given.

1.4 Claim of Contributions

The following are some of the contributions of this thesis:

- the modelling of non-normal modal logic systems as LSLTV, defining a modularized sequent calculus to this class of logics;
- the definition of fusion of LSLTV as a categorical operation, amplifying not only the class of logics allowed to use this kind of combination but also allowing different sets of shared connectives and operators between the component logics;
- the preservation of soundness and completeness in the defined fusion of LSLTV, where completeness requires only mild conditions to holds;
- the fusion of non-normal modal logics: not only fusion between two non-normal modal logics but also the fusion between a normal and a non-normal system.

Chapter 2

Logic Systems Labelled with Truth Values

This chapter presents the framework used to describe logic systems throughout this thesis. The idea is to label logic systems by truth values as introduced in [MSSV04]. The Logic Systems Labelled with Truth Values (LSLTV) introduced in that work are herein reshaped in order to be used to present a wider class of logics. Observe that labelling by truth values allows, when in the context of a modal system, that the denotation of terms is a *range* of worlds instead of a single one as in [Vig00, Rus96]. The results in this chapter are proved in a similar way to the corresponding ones in [MSSV04].

The chapter is divided in five sections, Section 2.1 concentrates on the language of LSLTV, concerned with assertions about formulas and terms (that represent truth values). In Section 2.2 it is introduced a Gentzen calculus of LSLTV, and derivation and consequence are defined. In Section 2.3 general metatheorems are shown to hold in this framework, adapting some results of [CRS08b] to the labelled case. In Section 2.4, about the two-sorted algebraic semantics of LSLTV, the basic concepts of denotation, satisfaction and entailment are defined. With these three main pieces defined: language, deduction and semantics, Section 2.5 about logic systems comes naturally, and results about soundness and completeness are showed.

Some simple examples are given along this chapter. In Chapter 3 a whole plethora of logic systems is described in this framework.

2.1 Language

The objective of this section is to introduce the language of the assertions of Logic Systems Labelled with Truth Values. Basically, the language is composed of expressions of the form $\theta \leq \varphi$ expressing that the truth value associated with the term θ is less than or equal to the denotation of formula φ. As it will be seen, in logics endowed with a Kripke semantics, truth values can be seen as sets of worlds.

The language is built from a signature where connectives, operators and sets of variables are specified.

Definition 2.1.1. A *signature* is a tuple $\Sigma = \langle C, O, \mathsf{P}, X, Y, Z \rangle$ where:

- $C = \{C_k : k \in \mathbb{N}\}$ where each C_k is a countable set;
- $O = \{O_k : k \in \mathbb{N}\}$ such that $\bot, \top \in O_0$ where each O_k is a countable set;
- $\mathsf{P} = \{\mathsf{P}_k : k \in \mathbb{N}^+\}$ where each P_k is a countable set;
- X, Y and Z are countable sets.

All these sets are pairwise disjoints. ∎

The elements of each C_k are known as (formula) constructors or connectives of arity k. Those of each O_k are known as (truth value) operators of arity k. Moreover the elements of each P_k are relations (between terms) of arity k. There are three kinds of variables: truth value unbound variables (usually represented as $\mathbf{x}$, element of X), truth value bound variables (usually represented as $\mathbf{y}$, element of Y) and formula unbound variables (usually represented as $\mathbf{z}$, element of Z). Sets X and Z are needed by the completeness proof while Y is useful when writing rules with fresh variables.

Example 2.1.2. A *signature for the classical propositional logic* is

$$\Sigma_P = \langle C, O, \mathsf{P}, X, Y, Z \rangle$$

where:

- $C_0 = \{\mathbf{t}, \mathbf{f}\} \cup \{\mathbf{p}_i : i \in \mathbb{N}\}$, $C_1 = \{\neg\}$, $C_2 = \{\wedge, \vee, \supset\}$, $C_n = \varnothing$ for all $n \geq 3$;
- $O_0 = \{\top, \bot\}$, $O_n = \varnothing$ for all $n \geq 1$;
- $\mathsf{P}_1 = \{\Omega\}$, $\mathsf{P}_2 = \{\sqsubseteq\}$, $\mathsf{P}_n = \varnothing$ for all $n \geq 3$; and

- X, Y, Z are disjoint countable sets of variables. ■

The next three definitions show how to construct formulas, terms and assertions. Formulas and terms are defined as expected. For that, assume given the following three sets $\Xi = \{\xi_i : i \in \mathbb{N}\}$, $\mathrm{T} = \{\tau_i : i \in \mathbb{N}\}$ and $\mathcal{G} = \{\Gamma_i : i \in \mathbb{N}\}$, whose elements are called meta-variables of formulas, terms and bags of assertions, respectively.

Definition 2.1.3. The set $F(\Sigma)$ of *(schema) simple formulas* over Σ is inductively defined as follows:

i. $\xi_i \in F(\Sigma)$ for every $i \in \mathbb{N}$;

ii. $\mathbf{z} \in F(\Sigma)$ for every $\mathbf{z} \in Z$; and

iii. $c(\varphi_1, ..., \varphi_k) \in F(\Sigma)$ whenever $c \in C_k$ and $\varphi_1, \ldots, \varphi_k \in F(\Sigma)$.

The set $gF(\Sigma)$ of *ground simple formulas* is composed of the elements in $F(\Sigma)$ without meta-variables and $cF(\Sigma)$, the set of *closed simple formulas*, is the set of elements of $gF(\Sigma)$ without variables. ■

Given the signature for classical propositional logic in Example 2.1.2 the following formulas can be constructed: $\xi_1 \supset \mathbf{p}_3$, $\xi_2 \supset \neg\, \xi_1$, $\mathbf{z}_1 \vee (\neg\, \mathbf{p}_1 \supset \mathbf{z}_2)$ and $\neg\, \mathbf{f}$. Only the last two belong to $gF(\Sigma)$ and only $\neg\, \mathbf{f}$ belongs to $cF(\Sigma)$.

As mentioned before, terms are constructed similarly to formulas. For the labelled logic systems considered in this work, terms are seen as labels, representing truth values.

Definition 2.1.4. The set $T(\Sigma)$ of *(schema) terms* over Σ is inductively defined as follows:

i. $\tau_i \in T(\Sigma)$ for every $i \in \mathbb{N}$;

ii. $\mathbf{x} \in T(\Sigma)$ for every $\mathbf{x} \in X$;

iii. $\mathbf{y} \in T(\Sigma)$ for every $\mathbf{y} \in Y$;

iv. $o(\theta_1, ..., \theta_k) \in T(\Sigma)$ whenever $o \in O_k$ and $\theta_1, \ldots, \theta_k \in T(\Sigma)$; and

v. $\#\varphi \in T(\Sigma)$ whenever $\varphi \in F(\Sigma)$.

The set $gT(\Sigma)$ of *ground terms* is composed of the elements in $T(\Sigma)$ without meta-variables and $cT(\Sigma)$, the set of *closed terms*, is the set of elements of $gT(\Sigma)$ without variables. ■

For instance, assuming that $\mathbf{o} \in O_2$ the following belong to $T(\Sigma)$: τ_3, $\#\xi_1$ and $\mathbf{o}(\mathbf{y}_2, \mathbf{x}_1)$. Among them, only $\mathbf{o}(\mathbf{y}_2, \mathbf{x}_1) \in gT(\Sigma)$ and no one belongs to $cT(\Sigma)$. The intended purpose of $\#\varphi$ is to represent syntactically the truth value associated to the formula φ.

Reasoning in the logic systems considered herein is done on assertions that can be a labelled formula or a relation between terms as defined in the sequel.

Definition 2.1.5. The set $A(\Sigma)$ of *(schema) assertions* over Σ is composed of expressions of the following forms:

- $\theta \leq \varphi$ and $\theta \not\leq \varphi$ (positive and negative labelled formula, respectively) for each $\theta \in T(\Sigma)$ and $\varphi \in F(\Sigma)$,
- $\varrho(\theta_1, \ldots, \theta_n)$ and $\not\varrho(\theta_1, \ldots, \theta_n)$ (positive and negative relation between terms, respectively) with $\varrho \in \mathbb{P}_n$ and $\theta_1, \ldots, \theta_n \in T(\Sigma)$.

The set $gA(\Sigma)$ of *ground assertions* is composed of the elements in $A(\Sigma)$ without meta-variables and $cA(\Sigma)$, the set of *closed assertions*, is the set of elements of $gA(\Sigma)$ without variables. ■

The notion of *conjugate* $\overline{\delta}$ is defined as follows:

- $\overline{\theta \leq \varphi}$ is $\theta \not\leq \varphi$;
- $\overline{\theta \not\leq \varphi}$ is $\theta \leq \varphi$;
- $\overline{\varrho(\theta_1, \ldots, \varrho_n)}$ is $\not\varrho(\theta_1, \ldots, \theta_n)$;
- $\overline{\not\varrho(\theta_1, \ldots, \theta_n)}$ is $\varrho(\theta_1, \ldots, \varrho_n)$.

In the sequel the unary relation $\Omega \in \mathbb{P}_1$ is used to express that a truth value is atomic (there is no truth value strictly smaller than it besides falsum) and the binary relation $\sqsubseteq \in \mathbb{P}_2$ to compare two truth values. The Ω relation is needed to establish the notion of local derivation, for instance. Both relations are used in order sequent calculus (Definition 2.2.7).

For instance, in the context of Σ_P it can be considered as assertions expressions as $\Omega\mathbf{y}_1$, $\mathbf{t} \leq \mathbf{y}_2$, and $\top \not\leq \xi_1 \wedge \xi_2$.

In the sequel, $\mathcal{B}_f(U)$ denotes the set of all finite bags of elements in the set U.

Substitutions are maps over meta-variables as is described in the definition below. They are used in derivations to assign concrete values to meta-variables.

Definition 2.1.6. A *(schema) substitution* over $A(\Sigma)$ is a map σ such that, for all $i \in \mathbb{N}$:

i. $\sigma(\xi_i) \in F(\Sigma)$;

ii. $\sigma(\tau_i) \in T(\Sigma)$; and

iii. $\sigma(\Gamma_i) \in \mathcal{B}_f(A(\Sigma) \cup \mathcal{G})$.

The set of (schema) substitutions over $A(\Sigma)$ is denoted by $Sbs(A(\Sigma))$.

A *ground substitution* over $A(\Sigma)$ is a schema substitution ρ such that, for all $i \in \mathbb{N}$:

i. $\rho(\xi_i) \in gF(\Sigma)$;

ii. $\rho(\tau_i) \in gT(\Sigma)$; and

iii. $\rho(\Gamma_i) \in \mathcal{B}_f(gA(\Sigma))$.

The set of ground substitutions over $A(\Sigma)$ is denoted by $gSbs(A(\Sigma))$. ∎

In what follows, being ω a substitution, $\Gamma\omega$ denotes $\omega(\Gamma)$ and the same applies for formulas, and truth values terms.

2.2 Deduction

This work will consider labelled logic systems adopting a Gentzen calculus for presenting the notion of deduction. Sequent calculi were introduced by Gerhard Gentzen in [Gen69] with the aim "to set up a formalism that reflects as accurate as possible the actual logic reasoning involved in mathematical proofs". In contrast, for instance, with Hilbert systems, the process of proving a theorem or a deduction is simpler and more natural.

Definition 2.2.1. A *sequent* over a set of assertions $A(\Sigma)$ is a pair $s = \langle \Delta_1, \Delta_2 \rangle$, written $\Delta_1 \to \Delta_2$, where $\Delta_1, \Delta_2 \in \mathcal{B}_f(A(\Sigma) \cup \mathcal{G})$.[1] ∎

[1] Remember that the elements of $\mathcal{B}_f(U)$ are *finite* bags of elements of U.

A sequent, described as

$$\vartheta_1, \ldots, \vartheta_n \rightarrow \zeta_1, \ldots, \zeta_k$$

with $\vartheta_1, \ldots, \vartheta_n, \zeta_1, \ldots, \zeta_k$ being assertions, means intuitively that the union of the assertions on the left side has at least one of the assertions on the right side as a consequence (that is, the disjunction of the right-side assertions is a consequence of the conjunction of the left-side assertions). If $n = 0$ then the left side is seen as $\top$ and if $k = 0$, the right side is seen as $\bot$.

Sometimes application of rules in derivations should be restricted by constraints. Here, constraints are presented in the form of provisos, that is, restrictions on substitutions.

Definition 2.2.2. A *(local) proviso* over $A(\Sigma)$ is a map $\pi : gSbs(A(\Sigma)) \rightarrow \{0, 1\}$.

The *unit proviso* **up** is such that $\mathbf{up}(\rho) = 1$ for every $\rho \in gSbs(A(\Sigma))$.

The *zero proviso* **zp** is such that $\mathbf{zp}(\rho) = 0$ for every $\rho \in gSbs(A(\Sigma))$. ■

Given two provisos π and π', their intersection is the proviso $(\pi \cap \pi')$ such that $(\pi \cap \pi')(\rho) = \pi(\rho) \times \pi'(\rho)$. The expression $\pi \subseteq \pi'$ denotes that $\pi(\rho) \leq \pi'(\rho)$ for each ground substitution ρ.

Given a schema substitution σ and a proviso π, the proviso $(\pi\sigma)$ is defined as $(\pi\sigma)(\rho) = \pi(\sigma\rho)$ for every $\rho \in gSbs(A(\Sigma))$. Notice that, for a ground substitution ρ, $(\pi\rho)$ is either **up** or **zp**.

In this work the following two provisos are needed:

- $(\tau_k : \mathbf{y})(\rho) = 1$ iff $\rho(\tau_k) \in Y$, and
- $(\tau_k \notin \Delta)(\rho) = 1$ iff $\rho(\tau_k)$ does not occur in $\Delta\rho$.

The first proviso only allows ground substitutions where τ_k is replaced by a truth value bound variable. The other proviso accepts a ground substitution ρ only if τ_k is replaced by a ground term not occurring in $\Delta\rho$.

Definition 2.2.3. A *rule* over $A(\Sigma)$ is a triple $r = \langle\{s_1, ..., s_p\}, s, \pi\rangle$ written

$$\frac{s_1 \quad \ldots \quad s_p}{s} \lhd \pi$$

where $s_1, ..., s_p, s$ are sequents over $A(\Sigma)$ and π is a proviso over $A(\Sigma)$. ■

The unit proviso **up** is usually omitted. Given a substitution σ and a rule r both over $A(\Sigma)$, the instance of r by σ

$$\frac{s_1\sigma \quad \dots \quad s_p\sigma}{s\sigma} \lhd \pi\sigma$$

is denoted by $r\sigma$ where, being s a sequent $\langle \Delta_1, \Delta_2 \rangle$, $s\sigma$ is the instance $\langle \Delta_1\sigma, \Delta_2\sigma \rangle$.

Definition 2.2.4. A rule is said to be *endowed with a persistent proviso* if its proviso does not change value when the context is enriched with closed assertions. ■

In this work, usually the context of a rule are the bags of assertions Γ_1 and Γ_2. The "fresh bound variable" proviso, with the form

$$\tau_2 : \mathbf{y}; \tau_2 \not\in \tau_1, \Gamma_1, \Gamma_2,$$

is persistent in every rule. Indeed, for every ground substitution ρ, it holds

$$(\tau_2 : \mathbf{y} \cap \tau_2 \not\in \tau_1, \Gamma_1, \Gamma_2)(\rho) = (\tau_2 : \mathbf{y} \cap \tau_2 \not\in \tau_1, \Gamma_1, \Gamma_2, \delta)(\rho)$$

as long as δ is a closed expression. Clearly, every rule endowed with the unit proviso **up** is endowed with a persistent proviso.

Definition 2.2.5. A *(sequent) calculus* is a pair $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ where Σ is a signature and $\mathcal{R}$ is a finite set of rules over $A(\Sigma)$. ■

Definition 2.2.6. A sequent calculus $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ is *structural* if $\mathcal{R}$ contains the axioms, weakening, contraction, conjugation and cut rules presented in Table 2.1. ■

Structural rules in Table 2.1 have the suffix 'F' when its main assertion is a labelled formula and 'ρ' when its main assertion is about the relation $\rho \in \mathbb{P}$. 'Ax' rules are the axioms and there also is one cut rule for each type of assertion. There are the weakening rules (Lw and Rw), the contraction rules (Lc and Rc) and the rules about the conjugation of the main assertion (Lxi, Lxe, Rxi and Rxe).

Definition 2.2.7. An *order sequent calculus* $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ is a structural sequent calculus such that $\Omega \in \mathbb{P}_1$, $\sqsubseteq \in \mathbb{P}_2$ and $\mathcal{R}$ has the rules presented in Table 2.2. ■

The symbol $\mho$ is used instead of $\not\Omega$ and $\sqsubseteq \langle t_1, t_2 \rangle$ is written $t_1 \sqsubseteq t_2$. The intended meaning of Ωt is that t is atomic, that is, there is no term strictly smaller than it besides falsum. Moreover $\sqsubseteq$ is intended to be the comparison between terms establishing an order of truth values: $t_1 \sqsubseteq t_2$ is read as "(the denotation of) the truth value t_1 is less than or equal to (the denotation of) the truth value t_2".

AxF	$\dfrac{}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$	Axϱ	$\dfrac{}{\varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2, \varrho(\tau_1,\ldots,\tau_n)}$
LwF	$\dfrac{\Gamma_1 \to \Gamma_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RwF	$\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$
Lwϱ	$\dfrac{\Gamma_1 \to \Gamma_2}{\varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}$	Rwϱ	$\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \varrho(\tau_1,\ldots,\tau_n)}$
LcF	$\dfrac{\tau_1 \leq \xi_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RcF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$
Lcϱ	$\dfrac{\varrho(\tau_1,\ldots,\tau_n), \varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}{\varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}$	Rcϱ	$\dfrac{\Gamma_1 \to \Gamma_2, \varrho(\tau_1,\ldots,\tau_n), \varrho(\tau_1,\ldots,\tau_n)}{\Gamma_1 \to \Gamma_2, \varrho(\tau_1,\ldots,\tau_n)}$
LxiF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2}$	RxiF	$\dfrac{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}$
Lxiϱ	$\dfrac{\Gamma_1 \to \Gamma_2, \varrho(\tau_1,\ldots,\tau_n)}{\not\varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}$	Rxiϱ	$\dfrac{\varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \not\varrho(\tau_1,\ldots,\tau_n)}$
LxeF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \not\leq \xi_1}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$	RxeF	$\dfrac{\tau_1 \not\leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$
Lxeϱ	$\dfrac{\Gamma_1 \to \Gamma_2, \not\varrho(\tau_1,\ldots,\tau_n)}{\varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}$	Rxeϱ	$\dfrac{\not\varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \varrho(\tau_1,\ldots,\tau_n)}$
cutF	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$		
cutϱ	$\dfrac{\Gamma_1 \to \Gamma_2, \varrho(\tau_1,\ldots,\tau_n) \quad \varrho(\tau_1,\ldots,\tau_n), \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$		

Table 2.1: Structural rules

Most of these rules come from the intuition that the truth values are subsets of worlds in a Kripke semantics [Kri63]. Notice also that the rule $\Omega\top$ allows the passage from local reasoning to global reasoning while rules RgenF and LgenF can be seen as the passage from an atomic truth value τ_2 of a formula to a higher truth value τ_1 such that τ_2 belongs to τ_1.

Example 2.2.8. The order sequent calculus $\mathcal{C}_P = \langle \Sigma_P, \mathcal{R}_P \rangle$ where Σ_P is the signature in Example 2.1.2 and $\mathcal{R}_P$ contains the rules presented in Table 2.3 and the rule

$$\mathbf{1} \quad \overline{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \top \sqsubseteq \tau_1}$$

is a calculus for the classical propositional logic. ∎

It is worthwhile to explain the **1** rule: in classical logic it is sufficient to consider the truth values $\top$ and $\bot$, so it can be assumed that any atomic truth value τ_1 is actually $\top$.

$$\Omega\top \quad \frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \top \leq \xi_1} \lhd \tau_1 : \mathbf{y}_1, \tau_1 \notin \Gamma_1, \Gamma_2$$

$$\Omega \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot \quad \Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1} \lhd \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$$

$$\Omega\bot \quad \frac{\Gamma_1 \to \Gamma_2, \Omega\tau_1}{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot} \qquad \top \quad \frac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \top}$$

$$\text{L}\# \quad \frac{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \#\xi_1, \Gamma_1 \to \Gamma_2} \qquad \text{R}\# \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \#\xi_1}$$

$$\bot\text{T} \quad \frac{}{\Gamma_1 \to \Gamma_2, \bot \sqsubseteq \tau_1} \qquad \bot\text{F} \quad \frac{}{\Gamma_1 \to \Gamma_2, \bot \leq \xi_1}$$

$$\text{cons} \quad \frac{}{\Gamma_1 \to \Gamma_2, \top \not\sqsubseteq \bot} \qquad \text{ref} \quad \frac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_1}$$

$$\text{Lasym} \quad \frac{\Omega\tau_1, \Omega\tau_2, \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2} \qquad \text{Rasym} \quad \frac{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1}$$

$$\text{transT} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_3}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_3}$$

$$\text{transF} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Gamma_1 \to \Gamma_2, \tau_2 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}$$

$$\text{LgenT} \quad \frac{\Omega\tau_2, \tau_2 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1 \quad \tau_1 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2, \Omega\tau_2}{\tau_1 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2}$$

$$\text{RgenT} \quad \frac{\Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_3}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_3} \lhd \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \tau_3, \Gamma_1, \Gamma_2$$

$$\text{LgenF} \quad \frac{\Omega\tau_2, \tau_2 \leq \xi_1, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1 \quad \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \Omega\tau_2}{\tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}$$

$$\text{RgenF} \quad \frac{\Omega\tau_2, \tau_2 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_2 \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1} \lhd \tau_2 : \mathbf{y}, \tau_2 \notin \tau_1, \Gamma_1, \Gamma_2$$

Table 2.2: Order rules

The next notion of this section is derivation. It is a sequence of steps where each step is either an instance of an axiom or a hypothesis or results from the application of an instance of some rule of the calculus.

Definition 2.2.9. Given a sequent calculus $\mathcal{C}$, a sequent s' is said to be *derived* from a set S of sequents with proviso π, written

$$S \vdash_{\mathcal{C}} s' \lhd \pi,$$

if there is a *derivation sequence* $(d_1, \pi_1)...(d_n, \pi_n)$ such that: d_1 is s' and $\pi \subseteq \pi_1$; and for every $i = 1, ..., n$:

1. either $d_i \in S$ and π_i is **up**;
2. or there is an assertion that occurs in both sides of d_i and π_i is **up**;

Lf	$\dfrac{\tau_1 \sqsubseteq \bot, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \mathbf{f}, \Gamma_1 \to \Gamma_2}$	**Rf**	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \bot}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \mathbf{f}}$
Lt	$\dfrac{\tau_1 \sqsubseteq \top, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \mathbf{t}, \Gamma_1 \to \Gamma_2}$	**Rt**	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \top}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \mathbf{t}}$
L$\wedge$	$\dfrac{\tau_1 \leq \xi_1, \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2}{\tau_1 \leq (\xi_1 \wedge \xi_2), \Gamma_1 \to \Gamma_2}$	R$\wedge$	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq (\xi_1 \wedge \xi_2)}$
L$\neg$	$\dfrac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Omega\tau_1, \tau_1 \leq (\neg\xi_1), \Gamma_1 \to \Gamma_2}$	R$\neg$	$\dfrac{\Omega\tau_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq (\neg\xi_1)}$
L$\supset$	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \quad \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2}{\tau_1 \leq (\xi_1 \supset \xi_2), \Gamma_1 \to \Gamma_2}$	R$\supset$	$\dfrac{\Omega\tau_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq (\xi_1 \supset \xi_2)}$
L$\vee$	$\dfrac{\Omega\tau_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2 \quad \Omega\tau_1, \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \tau_1 \leq (\xi_1 \vee \xi_2), \Gamma_1 \to \Gamma_2}$	R$\vee$	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1, \tau_1 \leq \xi_2}{\Gamma_1 \to \Gamma_2, \tau_1 \leq (\xi_1 \vee \xi_2)}$

Table 2.3: Rules for classical connectives

3. or there are $r \in \mathcal{R}, \rho \in Sbs(A(\Sigma)), p \in \mathbb{N}$ and $i_1, ..., i_p \in \{i+1, ..., n\}$ such that $r\rho = \dfrac{d_{i_1} \quad \ldots \quad d_{i_p}}{d_i} \lhd \pi'$ and $\pi_i = \pi' \cap \pi_{i_1} \cap \ldots \cap \pi_{i_p}$. ■

The result that follows shows basic properties of $\vdash$: it is projective, finitary, extensive, monotonic and idempotent. However these properties include provisos whenever needed. Notice that the projective property is not common in the usual consequence relations but they seem natural when dealing with provisos.

Proposition 2.2.10. For any sequent calculus $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$, the relation $\vdash$ is:

projective: if $S \vdash_{\mathcal{C}} s' \lhd \pi$ and $\pi' \subseteq \pi$ then $S \vdash_{\mathcal{C}} s' \lhd \pi'$;

finitary: if $S \vdash_{\mathcal{C}} s' \lhd \pi$ then there is a finite $S_1 \subseteq S$ such that $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$;

extensive: $S \vdash_{\mathcal{C}} s$ for each sequent $s \in S$;

monotonic: if $S \subseteq S_1$ and $S \vdash_{\mathcal{C}} s' \lhd \pi$ then $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$;

idempotent: if $S_1 \vdash_{\mathcal{C}} s \lhd \pi_s$ for each s in a finite set S of sequents and $S \vdash_{\mathcal{C}} s' \lhd \pi$ then $S_1 \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.

Proof. Indeed:

projective: Suppose $S \vdash_{\mathcal{C}} s' \lhd \pi$ with a derivation sequence $d = (d_1, \pi_1) \ldots (d_n, \pi_n)$, so by Definition 2.2.9 $d_1 = s'$ and $\pi \subseteq \pi_1$. Since $\pi' \subseteq \pi$, by transitivity of $\subseteq$, $\pi' \subseteq \pi_1$, then $S \vdash_{\mathcal{C}} s' \lhd \pi'$.

finitary: Suppose $S \vdash_{\mathcal{C}} s' \lhd \pi$ with a derivation sequence $d = (d_1, \pi_1) \ldots (d_n, \pi_n)$. Define inductively the sequence $S'_0, \ldots, S'_n$ as follows:

$$S'_0 = \varnothing$$
$$S'_i = \begin{cases} S'_{i-1} \cup \{d_i\} & \text{if } d_i \in S \\ S'_{i-1} & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n$. Take $S_1 = S'_n$.

extensive: Directly from Definition 2.2.9 item 1.

monotonic: Suppose $S \vdash_{\mathcal{C}} s' \lhd \pi$ with derivation sequence $d = (d_1, \pi_1) \ldots (d_n, \pi_n)$. Thus the same derivation can be used to show that $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$. This is proved by induction on the length of the derivation.

Base: $d = (d_1.\pi_1)$. Then (d_1, π_1) is such that $d_1 = s'$ and

1. either $d_1 \in S$ and $\pi_1 = \mathbf{up}$. So $\pi \subseteq \mathbf{up}$ and, since $S \subseteq S_1$, $d_1 \in S_1$. Therefore $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$;
2. or there is an assertion that occurs in both sides of d_1 and $\pi_1 = \mathbf{up}$. So $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$;
3. or there are $r \in \mathcal{R}$ and $\rho \in Sbs(A(\Sigma))$ such that $r\rho = \langle \varnothing, d_1, \pi_1 \rangle$, so $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$.

Step: The induction hypothesis is that $(d_2, \pi_2) \ldots (d_n, \pi_n)$ is a derivation from the set S_1. Then (d_1, π_1) is such that $d_1 = s'$ and (the cases 1. and 2. have proofs similar to the base)

3. there are $r \in \mathcal{R}$, $\rho \in Sbs(A(\Sigma))$, $p \in \mathbb{N}$ and $i_1, \ldots, i_p \in \{2, \ldots, n\}$ such that
$$r\rho = \frac{d_{i_1} \quad \ldots \quad d_{i_p}}{d_1} \lhd \pi'$$
and $\pi_1 = \pi' \cap \pi_{i_1} \cap \ldots \cap \pi_{i_p}$. Since, for each d_{i_j} with $j = 1, \ldots, p$, $S_1 \vdash_{\mathcal{C}} d_{i_j}$ by induction hypothesis, then $S_1 \vdash_{\mathcal{C}} s' \lhd \pi$.

idempotent: Since $\vdash$ is projective, $S_1 \vdash_{\mathcal{C}} s \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$ for each $s \in S$ and $S \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$. Denote by $d_s = (d^s_{1_s}, \pi^s_{1_s}) \ldots (d^s_{n_s}, \pi^s_{n_s})$ a derivation sequence for $S_1 \vdash_{\mathcal{C}} s \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$ for each $s \in S$ and by $d = (d_1, \pi_1) \ldots (d_m, \pi_m)$ a derivation sequence for $S \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$. Let d^* be a derivation sequence obtained from d by replacing each (d_i, π_i) with $d_i \in S$ by the derivation sequence d_{d_i}. It is now showed by induction on the length of d that d^* is a derivation sequence for $S_1 \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.

Base: $d = (d_1, \pi_1)$. Then (d_1, π_1) is such that $d_1 = s'$ and

1. either $d_1 \in S$ and $\pi_1 = \mathbf{up}$. So $d^* = d_{d_1}$ is a derivation sequence for $S_1 \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.
2. or there is an assertion that occurs in both sides of d_1 and $\pi_1 = \mathbf{up}$. So $d^* = (d_1, \pi_1)$ is a derivation sequence for $S_1 \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.
3. or there are $r \in \mathcal{R}$ and $\rho \in Sbs(A(\Sigma))$ such that $r\rho = \langle \varnothing, d_1, \pi_1 \rangle$. So $d^* = (d_1, \pi_1)$ is a derivation sequence for $S_1 \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.

Step: The induction hypothesis is such that for any derivation sequence $(d_j, \pi_j) \ldots (d_m, \pi_m)$ from d with $j = 2, \ldots, m$ there is a derivation sequence $d_j^\bullet$ of $S_1 \vdash_{\mathcal{C}} d_j \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$. Then (d_1, π_1) is such that $d_1 = s'$ and (the cases 1. and 2. have proofs similar to the base)

3. there are $r \in \mathcal{R}$, $\rho \in Sbs(A(\Sigma))$, $p \in \mathbb{N}$ and $i_1, \ldots, i_p \in \{2, \ldots, m\}$ such that

$$r\rho = \frac{d_{i_1} \quad \ldots \quad d_{i_p}}{d_1} \lhd \pi'$$

and $\pi_1 = \pi' \cap \pi_{i_1} \cap \ldots \cap \pi_{i_p}$. So, using the induction hypotheses, $d^* = (d_1, \pi_1) d_{i_1}^\bullet \ldots d_{i_p}^\bullet$ is a derivation sequence for $S_1 \vdash_{\mathcal{C}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$.

□

The use of truth value as labels allows the distinction between local and global reasoning that is interesting to logics such modal logics, for instance.

Definition 2.2.11. In an order calculus $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ a formula φ is:

- *globally derived* from formulas $\psi_1, \ldots, \psi_k$, denoted by

$$\psi_1, \ldots, \psi_k \vdash_{\mathcal{C}}^{g} \varphi$$

  iff $\varnothing \vdash_{\mathcal{C}} \top \leq \psi_1, \ldots, \top \leq \psi_k \to \top \leq \varphi$;

- *locally derived* from formulas $\psi_1, \ldots, \psi_k$, denoted by

$$\psi_1, \ldots, \psi_k \vdash_{\mathcal{C}}^{l} \varphi$$

  iff $\varnothing \vdash_{\mathcal{C}} \Omega \mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \ldots, \mathbf{y}_1 \leq \psi_k \to \mathbf{y}_1 \leq \varphi$. ■

Notice that in the proof of a local or a global consequence the derivation does not use hypothesis.

Proposition 2.2.12. Let $\mathcal{C}$ be an order calculus, then the relation $\vdash^d$ is transitive both when $d = l$ or $d = g$.

Proof. Suppose without loss of generality that $\psi \vdash^l_{\mathcal{C}} \varphi$ and $\varphi \vdash^l_{\mathcal{C}} \gamma$. Denote by d_1 a derivation for $\vdash_{\mathcal{C}} \Omega\mathbf{y}, \mathbf{y} \leq \psi \to \mathbf{y} \leq \varphi$, and by d_2 a derivation for $\vdash_{\mathcal{C}} \Omega\mathbf{y}, \mathbf{y} \leq \varphi \to \mathbf{y} \leq \gamma$. Then,

1.	$\Omega\mathbf{y}, \mathbf{y} \leq \psi$	$\to$	$\mathbf{y} \leq \gamma$	cutF 2,3
2.	$\mathbf{y} \leq \varphi, \Omega\mathbf{y}, \mathbf{y} \leq \psi$	$\to$	$\mathbf{y} \leq \gamma$	LwF 4
3.	$\Omega\mathbf{y}, \mathbf{y} \leq \psi$	$\to$	$\mathbf{y} \leq \gamma, \mathbf{y} \leq \varphi$	RwF 5
4.	$\mathbf{y} \leq \varphi, \Omega\mathbf{y}$	$\to$	$\mathbf{y} \leq \gamma$	d_2
5.	$\Omega\mathbf{y}, \mathbf{y} \leq \psi$	$\to$	$\mathbf{y} \leq \varphi$	d_1.

The thesis follows immediately for $\vdash^g_{\mathcal{C}}$ by applying the cut rule. □

2.3 Metatheorems

In this section metatheorems are investigated in two different approaches. Firstly it is established that three metatheorems hold in the context of structural sequent calculi. These results are originally presented in [MSSV04] and are used afterwards to obtain completeness results.

After that, general forms of metatheorems of deduction and modus ponens, using sets of metavariables, are defined. They are presented in the level of the sequent consequence relation $\vdash$ with special attention to $\vdash^l$ and $\vdash^g$, the local and global derivation relations, respectively. These definitions adapt similar ones in [CRS08b] for the calculus defined in this work.

The metatheorem of conjugation is the first to be presented. In what follows, $\overline{\Delta}$ denotes the bag of conjugates $\{\overline{\delta} : \delta \in \Delta\}$.

Theorem 2.3.1. Let $\mathcal{C}$ be a structural sequent calculus. Then, for every set S of ground sequents and every ground sequent $\Delta' \to \Delta''$:

$$S \vdash_{\mathcal{C}} \Delta' \to \Delta'' \text{ iff } S \vdash_{\mathcal{C}} \to \Delta'', \overline{\Delta'}.$$

Proof. Suppose $\Delta' = \{\delta'_1, \ldots, \delta'_n\}$ where each δ'_i for $i = 1, \ldots, n$ is a ground assertion. Given a derivation d for $S \vdash_{\mathcal{C}} \Delta' \to \Delta''$, a derivation of $\to \Delta'', \overline{\Delta'}$ is obtained by prefixing d with n applications of Rx rules, one for each $\delta'_i \in \Delta'$. On the other way, given a derivation for $S \vdash_{\mathcal{C}} \to \Delta'', \overline{\Delta'}$, a derivation for $S \vdash_{\mathcal{C}} \Delta' \to \Delta''$ is obtained by applying n times the Lx rules. □

Next the metatheorem of contradiction is proved to hold in structural sequent calculus. It states that if a calculus derives a bag of ground assertions and their conjugations, then it can derive any ground assertion.

Theorem 2.3.2. Let $\mathcal{C}$ be a structural sequent calculus. Then, for every set S of ground sequents and every ground sequent $\to \Delta$, if

$$\begin{cases} S \vdash_{\mathcal{C}} \to \Delta \\ S \vdash_{\mathcal{C}} \to \overline{\delta} \text{ for every } \delta \in \Delta \end{cases}$$

then $S \vdash_{\mathcal{C}} \to v$ for every ground assertion v.

Proof. Suppose without loss of generality that Δ is a set with two assertions δ_1 and δ_2. Denote by d_{δ_i} a derivation for $S \vdash_{\mathcal{C}} \to \overline{\delta_i}$ for each $\delta_i \in \Delta$, and by d_Δ a derivation for $S \vdash_{\mathcal{C}} \to \Delta$. Then,

1.		$\to$	v	cut 2,3
2.		$\to$	v, δ_1	cut 4,5
3.	δ_1	$\to$	v	Lx 10
4.		$\to$	v, δ_1, δ_2	Rw 6
5.	δ_2	$\to$	v, δ_1	Lx 7
6.		$\to$	δ_1, δ_2	d_Δ
7.		$\to$	$v, \delta_1, \overline{\delta_2}$	Rw 8
8.		$\to$	$\delta_1, \overline{\delta_2}$	Rw 9
9.		$\to$	$\overline{\delta_2}$	d_{δ_2}
10.		$\to$	$v, \overline{\delta_1}$	Rw 11
11.		$\to$	$\overline{\delta_1}$	d_{δ_1}

□

In the context of sequent calculi, it is expected two different metatheorems of deduction. This first one is defined over a set of sequents and the consequence relation. This metatheorem will be used, for instance, in the proof of completeness for logic systems based in this type of calculus.

Theorem 2.3.3. Let $\mathcal{C}$ be a structural sequent calculus with rules endowed with persistent provisos. Then, for every set S of ground sequents and closed sequent $\delta'_1, \ldots, \delta'_m \to \Delta''$:

$$S \vdash_{\mathcal{C}} \delta'_1, \ldots, \delta'_m \to \Delta'' \text{ iff } S, \to \delta'_1, \ldots, \to \delta'_m \vdash_{\mathcal{C}} \to \Delta''.$$

Proof. ($\Rightarrow$) Assume $S \vdash_{\mathcal{C}} \delta'_1, \ldots, \delta'_m \to \Delta''$ and let d be a derivation sequence for it. Then,

1.		$\to$	Δ''	cut 2,3
2.		$\to$	Δ'', δ'_1	Rw (several times) 4
3.	δ'_1	$\to$	Δ''	cut 5,6
4.		$\to$	δ'_1	hyp
5.	δ'_1	$\to$	Δ'', δ'_2	Rw (several times) 7
6.	δ'_2, δ'_1	$\to$	Δ''	cut 9,10
7.	δ'_1	$\to$	δ'_2	Lw 8
8.		$\to$	δ'_2	hyp
		$\vdots$		
	$\delta'_m, \dots, \delta'_1$	$\to$	Δ''	d

($\Leftarrow$) Assume $S, \to \delta'_1, \dots, \to \delta'_m \vdash_{\mathcal{C}} \to \Delta''$ and let $d = d_1, \dots, d_n$ be a derivation sequence for it. Changing each $d_i = \Gamma_1^i \to \Gamma_2^i$ to $d'_i = \delta'_1, \dots, \delta'_m, \Gamma_1^i \to \Gamma_2^i$ and in each $d_i = \to \delta'_j$ where the justification is "hyp", in d'_i the justification is replaced by "axiom", a derivation $S \vdash_{\mathcal{C}} \delta'_1, \dots, \delta'_m \to \Delta''$ is then build. The new sequence $d'_1, \dots, d'_n$ is actually a derivation since $\delta'_1, \dots, \delta'_m$ are closed assertions and therefore any proviso is still fulfilled (remembering that in all rules there are only persistent provisos). □

The metatheorem of deduction and the metatheorem of modus ponens are now defined in a different context, over a set of assertions and the sequent. In what follows, $A^{\Xi',\mathrm{T}'}(\Sigma)$ is the set of assertions generated from the sets of meta-variables Ξ' and T' for formulas and terms, respectively (instead of the sets Ξ and T).

Definition 2.3.4. A calculus $\mathcal{C}$ has the *metatheorem of deduction (MTD)* if there is a finite set of assertions $\Lambda \subseteq A^{\{\xi_1,\xi_2\},\{\tau\}}(\Sigma)$ such that:

$$\text{if } \vdash_{\mathcal{C}} \delta_1, \dots, \delta_n, t \leq \varphi_1 \to t \leq \varphi_2 \lhd \pi \text{ then } \vdash_{\mathcal{C}} \delta_1, \dots, \delta_n \to \Lambda(\varphi_1, \varphi_2, t) \lhd \pi$$

where $\delta_1, \dots, \delta_n \in A^{\Xi,\mathrm{T}}(\Sigma)$, $t \in T(\Sigma)$, $\varphi_1, \varphi_2 \in F(\Sigma)$ and $\Lambda(\varphi_1, \varphi_2, t)$ is obtained from Λ by substituting ξ_i by φ_i for $i = 1, 2$ and τ by t.

A calculus has the *metatheorem of modus ponens (MTMP)* if there is a finite set of assertions $\Lambda \subseteq A^{\{\xi_1,\xi_2\},\{\tau\}}(\Sigma)$ such that the converse holds. We refer to Λ as the *base set.* ■

Definition 2.3.5. A calculus $\mathcal{C}$ has *local metatheorem of deduction (l-MTD)* if the Definition 2.3.4 holds for $\delta_1 = \Omega\mathbf{y}$, $\delta_j = \mathbf{y} \leq \gamma_{j-1}$ for $j = 2, \dots, n$ and $t = \mathbf{y}$. The calculus $\mathcal{C}$ has *global metatheorem of deduction (g-MTD)* if $\delta_i = \top \leq \gamma_i$ for $i = 1, \dots, n$ and $t = \top$.

Similarly for *l-MTMP* and *g-MTMP*. ■

Example 2.3.6. The classical propositional calculus has l-MTD, g-MTD, l-MTMP and g-MTMP with $\Lambda = \{\tau \leq \xi_1 \supset \xi_2\}$.

The derivation to show the l-MTD is obtained simply by applying the R⊃ rule. For l-MTMP the derivation is shown below:

1.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \gamma_1, \ldots, \mathbf{y_1} \leq \gamma_n, \mathbf{y_1} \leq \varphi_1$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_2$	cutF 2,3
2.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \gamma_1, \ldots, \mathbf{y_1} \leq \gamma_n, \mathbf{y_1} \leq \varphi_1$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_2, \mathbf{y_1} \leq \varphi_1 \supset \varphi_2$	RwF 4
3.	$\mathbf{y_1} \leq \varphi_1 \supset \varphi_2, \Omega\mathbf{y_1}, \mathbf{y_1} \leq \gamma_1, \ldots, \mathbf{y_1} \leq \gamma_n, \mathbf{y_1} \leq \varphi_1$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_2$	L⊃ 5,6
4.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \gamma_1, \ldots, \mathbf{y_1} \leq \gamma_n, \mathbf{y_1} \leq \varphi_1$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_1 \supset \varphi_2$	LwF 7
5.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \gamma_1, \ldots, \mathbf{y_1} \leq \gamma_n, \mathbf{y_1} \leq \varphi_1$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_2, \mathbf{y_1} \leq \varphi_1$	axiom
6.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \varphi_2, \mathbf{y_1} \leq \gamma_1, \ldots, \mathbf{y_1} \leq \gamma_n, \mathbf{y_1} \leq \varphi_1$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_2$	axiom
7.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \gamma_1, \ldots, \mathbf{y_1} \leq \gamma_n$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_1 \supset \varphi_2$	hyp

The proof of the g-MTD is the derivation:

1.	$\top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_1 \supset \varphi_2$	RgenF 2
2.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_1 \supset \varphi_2$	R⊃ 3
3.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \varphi_1, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\mathbf{y_1} \leq \varphi_2$	transF 4,5
4.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \varphi_1, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \top$	axiom
5.	$\Omega\mathbf{y_1}, \mathbf{y_1} \leq \varphi_1, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2$	LgenF 6,7,8
6.	$\Omega\top, \top \leq \varphi_1, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2$	LwT 9
7.	$\Omega\top, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2, \top \sqsubseteq \mathbf{y_1}$	**1**

8.	$\mathbf{y_1} \leq \varphi_1, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2, \Omega\top$	Ω 11,12
9.	$\Omega\top, \Omega\mathbf{y_1}, \top \leq \varphi_1, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2$	LwΩ 10
10.	$\top \leq \varphi_1, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2$	hyp
11.	$\mathbf{y_1} \leq \varphi_1, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2, \top \not\sqsubseteq \bot$	cons
12.	$\Omega\mathbf{y_2}, \mathbf{y_2} \sqsubseteq \top, \mathbf{y_1} \leq \varphi_1, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2, \top \sqsubseteq \mathbf{y_2}$	**1**

And, finally, the g-MTMP comes from the following derivation:

1.	$\top \leq \gamma_1, \ldots, \top \leq \gamma_n, \top \leq \varphi_1$	$\rightarrow$	$\top \leq \varphi_2$	cutF 2,3
2.	$\top \leq \gamma_1, \ldots, \top \leq \gamma_n, \top \leq \varphi_1$	$\rightarrow$	$\top \leq \varphi_2, \top \leq \varphi_1 \supset \varphi_2$	Lw 4
3.	$\top \leq \varphi_1 \supset \varphi_2, \top \leq \gamma_1, \ldots,$ $\top \leq \gamma_n, \top \leq \varphi_1$	$\rightarrow$	$\top \leq \varphi_2$	L$\supset$ 6,7
4.	$\top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_2, \top \leq \varphi_1 \supset \varphi_2$	Rw 5
5.	$\top \leq \gamma_1, \ldots, \top \leq \gamma_n$	$\rightarrow$	$\top \leq \varphi_1 \supset \varphi_2$	hyp
6.	$\top \leq \gamma_1, \ldots, \top \leq \gamma_n, \top \leq \varphi_1$	$\rightarrow$	$\top \leq \varphi_2, \top \leq \varphi_1$	axiom
7.	$\top \leq \varphi_2, \top \leq \gamma_1, \ldots, \top \leq \gamma_n, \top \leq \varphi_1$	$\rightarrow$	$\top \leq \varphi_2$	axiom ■

2.4 Semantics

The semantics of LSLTV is based on a two-sorted algebra: a sort for truth values and another for denotations of formulas. This allows to encompass, in the framework of LSLTV, not only general Kripke structures as shown in Section 3.1 but also, for instance, the description of modal algebras. It is worthwhile to observe that the splitting in two sorts allows the representation of general Kripke structures instead of the standard ones.

Definition 2.4.1. Let Σ be a signature. A Σ-*algebra* is a triple $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}} \rangle$ where F and T are sets and $\cdot_{\mathbb{A}}$ is a map such that:

- $c_{\mathbb{A}} : F^k \rightarrow F$ for each $c \in C_k$;
- $o_{\mathbb{A}} : T^k \rightarrow T$ for each $o \in O_k$,
- $\#_{\mathbb{A}} : F \rightarrow T$;
- $\leq_{\mathbb{A}} \subseteq T \times F$;
- $\varrho_{\mathbb{A}} \subseteq T^k$ for each $\varrho \in \mathsf{P}_k$. ■

The notion of reduct of an algebra by a signature will be useful along this thesis, specially when defining fusion algebraically in Section 4.1.

Definition 2.4.2. Given a Σ-algebra $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}} \rangle$ and a signature Σ' such that $\Sigma' \subseteq \Sigma$, the *reduct of* $\mathbb{A}$ *by* Σ' is the Σ'-algebra $\mathbb{A}|_{\Sigma'} = \langle F, T, \cdot_{\mathbb{A}|_{\Sigma'}} \rangle$ such that:

- $c_{\mathbb{A}|_{\Sigma'}} = c_{\mathbb{A}}$ for all $c \in \Sigma'$,
- $o_{\mathbb{A}|_{\Sigma'}} = o_{\mathbb{A}}$ for all $o \in \Sigma'$,
- $\#_{\mathbb{A}|_{\Sigma'}} = \#_{\mathbb{A}}$,

- $\leq_{\mathbb{A}|_{\Sigma'}} = \leq_{\mathbb{A}}$, and
- $\varrho_{\mathbb{A}|_{\Sigma'}} = \varrho_{\mathbb{A}}$ for each $\varrho \in \mathsf{P}$. ■

In order to define the denotation of formulas and terms, the notion of assignment over a Σ-algebra is needed. Basically it specifies the meaning of the variables in the Σ-algebra.

Definition 2.4.3. An *unbound variable assignment* over $\mathbb{A}$ is a function α that maps each element of X to an element of T and each element of Z to an element of F. A *bound variable assignment* over $\mathbb{A}$ is a map β from Y to T. ■

The definition of denotation is split in two parts: the denotation of formulas (where only unbound variable assignments may be used) and the denotation of terms (where both kinds of assignments are needed).

Definition 2.4.4. The *denotation* of ground simple formulas at a Σ-algebra $\mathbb{A}$ for an unbound variable assignment α, is inductively defined in the following way:

- $[\![\mathbf{z}]\!]_{\mathbb{A}\alpha} = \alpha(z)$;
- $[\![c(\varphi_1, \ldots, \varphi_k)]\!]_{\mathbb{A}\alpha} = c_{\mathbb{A}}([\![\varphi_1]\!]_{\mathbb{A}\alpha}, \ldots, [\![\varphi_k]\!]_{\mathbb{A}\alpha})$.

The *denotation* at $\mathbb{A}$ for assignments α, β over $\mathbb{A}$ of ground terms is inductively defined in the following way:

- $[\![\mathbf{x}]\!]_{\mathbb{A}\alpha\beta} = \alpha(x)$;
- $[\![\mathbf{y}]\!]_{\mathbb{A}\alpha\beta} = \beta(y)$;
- $[\![o(\theta_1, \ldots, \theta_k)]\!]_{\mathbb{A}\alpha\beta} = o_{\mathbb{A}}([\![\theta_1]\!]_{\mathbb{A}\alpha\beta}, \ldots, [\![\theta_k]\!]_{\mathbb{A}\alpha\beta})$;
- $[\![\#\varphi]\!]_{\mathbb{A}\alpha\beta} = \#_{\mathbb{A}}([\![\varphi]\!]_{\mathbb{A}\alpha})$. ■

The definition of satisfaction of assertions and sequents can be given capitalizing on the denotation of terms and formulas presented above.

Definition 2.4.5. The *satisfaction* by $\mathbb{A}$ for α and β of ground assertions and sequents is defined as follows:

- $\mathbb{A}\alpha\beta \Vdash \varrho(\theta_1, \ldots, \theta_n)$ iff $\langle [\![\theta_1]\!]_{\mathbb{A}\alpha\beta}, \ldots, [\![\theta_n]\!]_{\mathbb{A}\alpha\beta} \rangle \in \varrho_{\mathbb{A}}$;

- $\mathbb{A}\alpha\beta \Vdash \not\varrho(\theta_1, \ldots, \theta_n)$ iff $\langle [\![\theta_1]\!]_{\mathbb{A}\alpha\beta}, \ldots, [\![\theta_n]\!]_{\mathbb{A}\alpha\beta}\rangle \notin \varrho_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \leq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}\alpha}\rangle \in \leq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \theta \not\leq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}\alpha}\rangle \notin \leq_{\mathbb{A}}$;
- $\mathbb{A}\alpha\beta \Vdash \Delta' \to \Delta''$ iff $\mathbb{A}\alpha\beta \Vdash \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$.

Futhermore, the *satisfaction by* $\mathbb{A}$ *for* α for ground assertions and sequents is such that for every bound variable assignment β over $\mathbb{A}$, $\mathbb{A}\alpha \Vdash \delta$ iff $\mathbb{A}\alpha\beta \Vdash \delta$ for δ being an assertion and, for a sequent $\Delta' \to \Delta''$, $\mathbb{A}\alpha \Vdash \Delta' \to \Delta''$ iff $\mathbb{A}\alpha\beta \Vdash \Delta' \to \Delta''$. ■

Entailment is now defined over a class of algebras.

Definition 2.4.6. Given a class $\mathcal{A}$ of Σ-algebras, a ground sequent s is $\mathcal{A}$-*entailed* by the ground sequents $s_1, \ldots, s_p$, written

$$s_1, \ldots, s_p \vDash_{\mathcal{A}} s,$$

iff, for each $\mathbb{A} \in \mathcal{A}$ and unbound variable assignment α over $\mathbb{A}$, $\mathbb{A}\alpha \Vdash s$ whenever $\mathbb{A}\alpha \Vdash s_i$ for every $i = 1, \ldots, p$.

A sequent s is $\mathcal{A}$-*entailed* by the sequents $s_1, \ldots, s_p$ with proviso π, written $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$, iff $s_1\rho, \ldots, s_p\rho \vDash_{\mathcal{A}} s\rho$ for every ground substitution ρ over $A(\Sigma)$ such that $\pi(\rho) = 1$. ■

Similarly to Proposition 2.2.10 where properties for $\vdash$ are proved, the following statement shows properties of $\vDash$.

Proposition 2.4.7. Given a class $\mathcal{A}$ of Σ-algebras, the relation $\vDash$ is:

projective: if $S \vDash_{\mathcal{A}} s' \lhd \pi$ and $\pi' \subseteq \pi$ then $S \vDash_{\mathcal{A}} s' \lhd \pi'$;

extensive: $S \vDash_{\mathcal{A}} s$ for each sequent $s \in S$;

monotonic: if $S \subseteq S_1$ and $S \vDash_{\mathcal{A}} s' \lhd \pi$ then $S_1 \vDash_{\mathcal{A}} s' \lhd \pi$;

idempotent: if $S_1 \vDash_{\mathcal{A}} s \lhd \pi_s$ for each s in a finite set S of sequents and $S \vDash_{\mathcal{A}} s' \lhd \pi$ then $S_1 \vDash_{\mathcal{A}} s' \lhd \pi \cap (\bigcap_{s\in S} \pi_s)$.

Proof. The first three properties are immediate to prove from the Definition 2.4.6. With respect to the last one, assume $S_1 \vDash_{\mathcal{A}} s \lhd \pi_s$ for each sequent $s \in S$ and $S \vDash_{\mathcal{A}} s' \lhd \pi$. Since $\vDash$ is projective, $S_1 \vDash_{\mathcal{A}} s \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$ for each s and $S \vDash_{\mathcal{A}} s' \lhd \pi \cap (\bigcap_{s \in S} \pi_s)$. So, for every ρ such that $\pi \cap (\bigcap_{s \in S} \pi_s)(\rho) = 1$, $S_1\rho \vDash_{\mathcal{A}} s\rho$ for each $s \in S$ and $S\rho \vDash_{\mathcal{A}} s'\rho$. Therefore, for every such ρ, every $\mathbb{A} \in \mathcal{A}$ and every unbound assignment α over $\mathbb{A}$: for every $s \in S, \mathbb{A}\alpha \Vdash s\rho$ whenever $\mathbb{A}\alpha \Vdash s_1\rho$ for every $s_1 \in S_1$; and $\mathbb{A}\alpha \Vdash s'\rho$ whenever $\mathbb{A}\alpha \Vdash s\rho$ for every $s \in S$. So, for every such ρ, every $\mathbb{A} \in \mathcal{A}$ and α, if $\mathbb{A}\alpha \Vdash s_1\rho$ for every $s_1 \in S_1$ then $\mathbb{A}\alpha \Vdash s'\rho$. □

Like in Definition 2.2.11, it is possible to distinguish between local and global entailment.

Definition 2.4.8. Let $\mathcal{A}$ be a class of Σ-algebras. A formula φ is *globally entailed* from formulas $\psi_1, \ldots, \psi_k$, denoted by $\psi_1, \ldots, \psi_k \vDash^g_{\mathcal{A}} \varphi$ iff $\vDash_{\mathcal{A}} \top \leq \psi_1, \ldots, \top \leq \psi_k \to \top \leq \varphi$. A formula φ is *locally entailed* from formulas $\psi_1, \ldots, \psi_k$, denoted by $\psi_1, \ldots, \psi_k \vDash^l_{\mathcal{A}} \varphi$ iff $\Omega \in \mathbb{P}_1$ and $\vDash_{\mathcal{A}} \Omega\mathbf{y_1}, \mathbf{y_1} \leq \psi_1, \ldots, \mathbf{y_1} \leq \psi_k \to \mathbf{y_1} \leq \varphi$. ■

Finally, as for the deductive consequence, entailment is also preserved by substitutions.

Proposition 2.4.9. Given a class $\mathcal{A}$ of Σ-algebras, for every substitution σ, if $S \vDash_{\mathcal{A}} s' \lhd \pi$ then $S\sigma \vDash_{\mathcal{A}} s'\sigma \lhd \pi\sigma$.

Proof. Observe that, if $S \vDash_{\mathcal{A}} s' \lhd \pi$ then for every ground substitution ρ' such that $\pi(\rho') = 1$ it happens that $S\rho' \vDash_{\mathcal{A}} s'\rho'$. Take an arbitrary ground substitution ρ such that $(\pi\sigma)(\rho) = 1$, then $\sigma\rho$ is ground and $\pi(\sigma\rho) = (\pi\sigma)\rho = 1$. Hence, using the hypothesis, $S(\sigma\rho) \vDash_{\mathcal{A}} s'(\sigma\rho)$, and so the thesis follows since for any substitutions $\delta(\sigma\rho) = (\delta\sigma)\rho$. □

The section finishes by introducing a definition needed to establish a connection between deduction and semantics.

Definition 2.4.10. A class of algebras $\mathcal{A}$ is called *appropriate* for the rule $r = \langle \{s_1, ..., s_p\}, s, \pi \rangle$ if $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$. Moreover an algebra $\mathbb{A}$ is said to be *appropriate for a rule* if so is the class $\{\mathbb{A}\}$. The class of algebras is *full* for a calculus $\langle \Sigma, \mathcal{R} \rangle$ if it is the class of all Σ-algebras that are appropriate for all rules in $\mathcal{R}$. ■

Given a calculus $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$, it will be denoted by app($\mathcal{C}$) the class of algebras appropriate for $\mathcal{C}$.

2.5 Logic Systems

The definition of logic system is now presented. A logic system puts together the different components of a logic: the syntax, the deductive system and the semantics.

Definition 2.5.1. A *logic system* is a triple $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ where $\langle \Sigma, \mathcal{R} \rangle$ is a sequent calculus and $\mathcal{A}$ is a class of Σ- algebras. ■

A logic system is said to be:

- *sound* if $\mathcal{A}$ is appropriate for each rule in $\mathcal{R}$;
- *full* if $\mathcal{A}$ is the class of all Σ-algebras that are appropriate for each rule in $\mathcal{R}$;
- *complete* if $s_1, \ldots, s_p \vdash_{\mathcal{R}} s$ whenever $s_1, \ldots, s_p \vDash_{\mathcal{A}} s$ for any closed sequents $s, s_1, \ldots, s_p$.

Corollary 2.5.2. Every full logic is sound.

The next definition presents the class of logic systems mostly used in this work, the relational LSLTV. Intuitively a relational LSLTV is a structural logic system where its semantics can be expressed in terms of "Kripke-like" models.

Definition 2.5.3. A *relational LSLTV* $\mathcal{L}_R$ is a tuple $\langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ such that:

- Σ contains Σ_P together with $\mathbf{I} \in O_1$ and $\mathbf{lb} \in O_2$;
- $\mathcal{R}$ contains the structural rules, the order rules, rules for constants and for each propositional connective, and rules $\mathbf{I}, \Omega\mathbf{I}, \mathbf{lb}1$ and $\mathbf{lb}2$ presented in Table 2.4;
- $\mathcal{A}$ is a class of Σ-algebras. ■

$$\mathbf{I} \quad \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{I}(\tau_1) \sqsubseteq \tau_1} \qquad \Omega\mathbf{I} \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot}{\Gamma_1 \to \Gamma_2, \Omega\mathbf{I}(\tau_1)}$$

$$\mathbf{lb}1 \quad \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{lb}(\tau_1, \tau_2) \sqsubseteq \tau_1} \qquad \mathbf{lb}2 \quad \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{lb}(\tau_1, \tau_2) \sqsubseteq \tau_2}$$

Table 2.4: Specific rules for operators

The operator $\mathbf{lb}(t_1, t_2)$ represents a lower bound of the terms t_1 and t_2, and $\mathbf{I}(t)$ represents an atomic truth value contained in the denotation of the term t. Therefore rules **I** and **ΩI** impose that $\mathbf{I}(t)$ represents an atomic truth value contained in the denotation of t, as long as t is not $\bot$.

It is important to notice that for a relational LSLTV $\langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ based in classical propositional logic, the rule **1** presented in Example 2.2.8 is *not* necessarily in $\mathcal{R}$. In fact, it is usually not in most of the typical relational logic systems since intuitively, it means that there are only two truth values in the system.

The following two subsections bring results from [MSSV04].

2.5.1 Soundness

This subsection starts by showing that Σ-algebras are suitable models for structural calculus.

Theorem 2.5.4. The class of all Σ-algebras is appropriate for every structural rule over $A(\Sigma)$.

Proof. The proof consists in verifying $s_1, \ldots, s_n \vDash_{\mathcal{A}} s$ for every structural rule $\langle \{s_1, \ldots, s_n\}, s, \pi \rangle$ in Table 2.1. Only the case of the rule RwF is shown since for the other rules the proof follows similarly. Taken any Σ-algebra $\mathbb{A}$, assignment α and ground substitution ρ, assume that $\mathbb{A}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho$. That is, for every assignment β, $\mathbb{A}\alpha\beta \Vdash \Gamma_1\rho \to \Gamma_2\rho$. Thus, for every β there is $\delta \in \Gamma_2\rho \cup \overline{\Gamma_1\rho}$ such that $\mathbb{A}\alpha\beta \Vdash \delta$. So, for every β it happens that there is $\delta \in \Gamma_2\rho \cup \{\tau_1\rho \leq \xi_1\rho\} \cup \overline{\Gamma_1\rho}$ such that $\mathbb{A}\alpha\beta \Vdash \delta$. Therefore, $\mathbb{A}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \leq \xi_1\rho$. □

Hence a general result for soundness can now be established.

Theorem 2.5.5. A logic system $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ is sound iff $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$ whenever $s_1, \ldots, s_p \vdash_{\mathcal{R}} s \lhd \pi$ for every sequents $s_1, \ldots, s_p, s$.

Proof. Indeed:

($\Leftarrow$) It is sufficient to show that $\mathcal{A}$ is appropriate for $\langle \Sigma, \mathcal{R} \rangle$. So, given a rule $r = \langle \{s'_1, \ldots, s'_p\}, s', \pi' \rangle \in \mathcal{R}$ it is straightfoward to build a derivation of $s'_1, \ldots, s'_p \vdash_{\mathcal{R}} s' \lhd \pi'$ and, so, by hypothesis, $s'_1, \ldots, s'_p \vDash_{\mathcal{A}} s' \lhd \pi'$. So, $\mathcal{A}$ is appropriate for r.

($\Rightarrow$) Assume that $\mathcal{L}$ is sound and $s_1, \ldots, s_p \vdash_{\mathcal{R}} s \lhd \pi$ with derivation $d = (d_1, \pi_1) \ldots (d_n, \pi_n)$. The proof of $s_1, \ldots, s_p \vDash_{\mathcal{A}} s \lhd \pi$ is by induction on the length of d. To show that $s_1, \ldots, s_p \vDash_{\mathcal{A}} d_1 \lhd \pi_1$ it have to be considered three cases that justify the derivation:

hypothesis By Proposition 2.4.7 $\models_{\mathcal{A}}$ is extensive and projective. So $s_1, \ldots, s_p \models_{\mathcal{A}} s_i \lhd \pi_1$ for any $i = 1, \ldots, p$.

axiom Obviously $\models_{\mathcal{A}} \delta \to \delta$ and by weakening proved in Theorem 2.5.4 $\models_{\mathcal{A}} \delta, \Gamma_1 \to \Gamma_2, \delta$. Therefore, by monotonicity and projection from Proposition 2.4.7, $s_1, \ldots, s_p \models_{\mathcal{A}} \delta, \Gamma_1 \to \Gamma_2, \delta \lhd \pi_1$.

rule Assume that $r \in \mathcal{R}$ was used with substitution σ for justifying d_1. Let

$$r\sigma = \frac{d_{i_1} \quad \ldots \quad d_{i_q}}{d_1} \lhd \pi'$$

with $i_1, \ldots, i_q \in \{2, \ldots, n\}$ and $\pi_1 \subseteq \pi' \cap \pi_{i_1} \cap \ldots \cap \pi_{i_q}$. Since $\models_{\mathcal{A}}$ is projective (Proposition 2.4.7), it is enough to show $s_1, \ldots, s_p \models_{\mathcal{A}} d_1 \lhd \pi' \cap \pi_{i_1} \cap \ldots \cap \pi_{i_q}$. Considering that $\mathcal{A}$ is appropriate for $\mathcal{R}$ and using Proposition 2.4.9 then $d_{i_1}, \ldots, d_{i_q} \models_{\mathcal{A}} d_1 \lhd \pi'$. On the other hand, by induction hypothesis, $s_1, \ldots, s_p \models_{\mathcal{A}} d_{i_j} \lhd \pi_{i_j}$ for $j = 1, \ldots, q$. Therefore, by idempotence shown in Proposition 2.4.7 the envisaged result is obtained.

□

2.5.2 Completeness

In this subsection it is proved that completeness holds under mild conditions in logic systems with structural calculus. The conditions are that the rules of the calculus must have persistent provisos. The theorem is proved using a variant of the Lindenbaum-Tarski technique. This variance is expected once the logic systems deal herein uses truth values as labels. First, it starts by defining a maximal consistent set and the syntactical algebra.

Definition 2.5.6. A set S of closed sequents over $A(\Sigma)$ is said to be *consistent* if for no closed assertion δ both $S \vdash \to \delta$ and $S \vdash \to \overline{\delta}$ hold. And it is said to be *maximal consistent* if for every closed assertion δ either $\to \delta \in S$ or $\to \overline{\delta} \in S$ but not both.■

Definition 2.5.7. Given a sequent calculus $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$ and a maximal consistent set S of closed sequents over $A(\Sigma)$, the *syntactic algebra* induced by $\mathcal{C}$ and S is the following Σ-algebra:

$$\mathbb{A}(\mathcal{C}, S) = \langle cF(\Sigma), cT(\Sigma), \cdot_{\mathbb{A}(\mathcal{C},S)} \rangle$$

where

- $c_{\mathbb{A}(\mathcal{C},S)} = \lambda f_1 \ldots f_k . c(f_1, \ldots, f_k)$;

- $o_{\mathbb{A}(\mathcal{C},S)} = \lambda t_1 \dots t_k . o(t_1, \dots, t_k)$;
- $\#_{\mathbb{A}(\mathcal{C},S)} = \lambda f . \# f$;
- $\langle \tau, \varphi \rangle \in \leq_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}}\!\to \tau \leq \varphi$;
- $\langle \tau_1, \dots, \tau_n \rangle \in \varrho_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}}\!\to \varrho(\tau_1, \dots, \tau_n)$. ■

Let $\varphi \in gF(\Sigma)$ and $\theta \in gT(\Sigma)$. Given an unbounded variable assignment α and a bounded variable assignment β both over a syntactic algebra $\mathbb{A}(\mathcal{C}, S)$, $\varphi\alpha$ denotes the closed simple formula obtained from φ by replacing each variable $z \in Z$ by $\alpha(z)$; and $\theta\alpha\beta$ denotes the closed term obtained from θ by replacing each variable $x \in X$ by $\alpha(x)$ and each variable $y \in Y$ by $\beta(y)$. This notation is extended to ground assertions and bags of ground assertions by identifying $\varphi\alpha\beta$ with $\varphi\alpha$.

Lemma 2.5.8. Let $\mathcal{C}$ be a structural calculus, S a maximal consistent set of closed sequents, α an unbound variable assignment and β a bound variable assignment (both over $\mathbb{A}(\mathcal{C}, S)$), φ a ground simple formula, θ a ground term and δ a ground assertion. Then:

i. $[\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \varphi\alpha$;

ii. $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \theta\alpha\beta$;

iii. $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \delta$ iff $S \vdash_{\mathcal{C}}\!\to \delta\alpha\beta$;

Proof. (i) By structural induction on φ.

- φ is $\mathbf{z}$.
  $[\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = [\![\mathbf{z}]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \alpha(\mathbf{z}) = \varphi\alpha$.
- φ is $c(\varphi_1, \dots, \varphi_n)$.
  $[\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = [\![c(\varphi_1, \dots, \varphi_n)]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = c([\![\varphi_1]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}, \dots, [\![\varphi_n]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} =$ (by induction hypothesis) $c(\varphi_1\alpha, \dots, \varphi_n\alpha) = \varphi\alpha$.

(ii) By structural induction on θ.

- θ is $\mathbf{x}$.
  $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = [\![\mathbf{x}]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \alpha(\mathbf{x}) = \theta\alpha\beta$.
- θ is $\mathbf{y}$.
  $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = [\![\mathbf{y}]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \beta(\mathbf{y}) = \theta\alpha\beta$.

- θ is $o(\theta_1, \ldots, \theta_n)$.

  $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = [\![o(\theta_1, \ldots, \theta_n)]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = o([\![\theta_1]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}, \ldots, [\![\theta_n]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}) =$ (by induction hypothesis) $o(\theta_1\alpha\beta, \ldots, \theta_n\alpha\beta) = \theta\alpha\beta$.

- θ is $\#\varphi$ with φ being a ground formula.

  $[\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = [\![\#\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} = \#[\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta} =$ by i. $\#\varphi\alpha = \theta\alpha\beta$.

(iii) There are four cases to analyze:

- δ is $\theta \leq \varphi$.

  Then $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \theta \leq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}\rangle \in \leq_{\mathbb{A}(\mathcal{C},S)}$ iff (by i. and ii.) $\langle \theta\alpha\beta, \varphi\alpha\rangle \in \leq_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \theta\alpha\beta \leq \varphi\alpha$ iff $S \vdash_{\mathcal{C}} \to (\theta \leq \varphi)\alpha\beta$.

- δ is $\rho(\theta_1, \ldots, \theta_n)$.

  Then $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \varrho(\theta_1, \ldots, \theta_n)$ iff $\langle [\![\theta_1]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}, \ldots, [\![\theta_n]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}\rangle \in \varrho_{\mathbb{A}(\mathcal{C},S)}$ iff (by ii.) $\langle \theta_1\alpha\beta, \ldots, \theta_n\alpha\beta\rangle \in \varrho_{\mathbb{A}(\mathcal{C},S)}$ iff $S \vdash_{\mathcal{C}} \to \varrho(\theta_1\alpha\beta, \ldots, \theta_n\alpha\beta)$ iff $S \vdash_{\mathcal{C}} \to (\varrho(\theta_1, \ldots, \theta_n))\alpha\beta$.

- δ is $\theta \not\leq \varphi$.

  Then $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \theta \not\leq \varphi$ iff $\langle [\![\theta]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}, [\![\varphi]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}\rangle \notin \leq_{\mathbb{A}(\mathcal{C},S)}$ iff (by i. and ii.) $\langle \theta\alpha\beta, \varphi\alpha\rangle \notin \leq_{\mathbb{A}(\mathcal{C},S)}$ iff $S \not\vdash_{\mathcal{C}} \to \theta\alpha\beta \leq \varphi\alpha$ iff (since S is maximal consistent) $S \vdash_{\mathcal{C}} \to \theta\alpha\beta \not\leq \varphi\alpha$ iff $S \vdash_{\mathcal{C}} \to (\theta \not\leq \varphi)\alpha\beta$.

- δ is $\not\varrho(\theta_1, \ldots, \theta_n)$.

  Then $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \not\varrho(\theta_1, \ldots, \theta_n)$ iff $\langle [\![\theta_1]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}, \ldots, [\![\theta_n]\!]_{\mathbb{A}(\mathcal{C},S)\alpha\beta}\rangle \notin \varrho_{\mathbb{A}(\mathcal{C},S)}$ iff (by ii.) $\langle \theta_1\alpha\beta, \ldots, \theta_n\alpha\beta\rangle \notin \varrho_{\mathbb{A}(\mathcal{C},S)}$ iff $S \not\vdash_{\mathcal{C}} \to \varrho(\theta_1\alpha\beta, \ldots, \theta_n\alpha\beta)$ iff (since S is maximal consistent) $S \vdash_{\mathcal{C}} \to \not\varrho(\theta_1\alpha\beta, \ldots, \theta_n\alpha\beta)$ iff $S \vdash_{\mathcal{C}} \to \not\varrho(\theta_1, \ldots, \theta_n)\alpha\beta$.

□

Item **iii.** of Lemma 2.5.8 is the fundamental lemma to establish the completeness since makes a correspondence between the satisfaction in the syntactical algebra and the deduction in the calculus for any ground sequent.

Lemma 2.5.9. Let $\mathcal{C}$ be a structural calculus, S a maximal consistent set of closed sequents, α an unbound variable assignment and β a bound variable assignment (both over $\mathbb{A}(\mathcal{C}, S)$) and $\Delta' \to \Delta''$ a ground sequent. Then:

$$\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \Delta' \to \Delta'' \text{ iff } S \vdash_{\mathcal{C}} \Delta'\alpha\beta \to \Delta''\alpha\beta.$$

Proof. Indeed, $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \Delta' \to \Delta''$ iff $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$ iff (by Lemma 2.5.8) $S \vdash_{\mathcal{C}} \to \delta\alpha\beta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$ iff $S \vdash_{\mathcal{C}} \to \delta$ for some $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$ iff (by the justification below) $S \vdash_{\mathcal{C}} \to \Delta''\alpha\beta, \overline{\Delta'}\alpha\beta$ iff (by Theorem 2.3.1) $S \vdash_{\mathcal{C}} \Delta'\alpha\beta \to \Delta''\alpha\beta$. It remains to explain that:

(1) If $S \vdash_{\mathcal{C}} \to \delta$ for some $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$ then $S \vdash_{\mathcal{C}} \to \Delta''\alpha\beta, \overline{\Delta'}\alpha\beta$. This follows by several applications of right weakening.

(2) If $S \vdash_{\mathcal{C}} \to \Delta''\alpha\beta, \overline{\Delta'}\alpha\beta$ then $S \vdash_{\mathcal{C}} \to \delta$ for some $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$. Indeed, otherwise, since S is maximal consistent, $S \vdash_{\mathcal{C}} \to \overline{\delta}$ for every $\delta \in \Delta''\alpha\beta \cup \overline{\Delta'}\alpha\beta$. Then, using Theorem 2.3.2, every closed assertion is derivable from S, therefore contradicting that S is consistent. □

Taken a sequent calculus and a class of syntactical algebras induced by it, it is now shown that these algebras are appropriate for this calculus.

Lemma 2.5.10. The class of all syntactic algebras induced by a sequent calculus is appropriate for it.

Proof. Let $r = \dfrac{s_1 \dots s_p}{s}$ be a ground instance of a (proper) rule of a sequent calculus $\mathcal{C}$. Let α be an arbitrary unbound variable assignment over a syntactic algebra $\mathbb{A}(\mathcal{C}, S)$. Assume that $\mathbb{A}(\mathcal{C}, S)\alpha \Vdash s_i$ for each $i = 1, \dots, p$. That is, for every bound variable assignment β, $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash s_i$. So, by Lemma 2.5.9, for every such β, $S \vdash_{\mathcal{C}} s_i\alpha\beta$ with derivation sequence $d^{s_i\alpha\beta}$, for each such i. Then, it is straightforward to build, for every pair α, β, a derivation sequence for $S \vdash_{\mathcal{C}} s\alpha\beta$ using rule r and those derivation sequences. Thus, again by Lemma 2.5.9, for every such β, $\mathbb{A}(\mathcal{C}, S)\alpha\beta \Vdash s$. That is, $\mathbb{A}(\mathcal{C}, S) \Vdash s$. □

The next lemma is crucial to prove completeness. It makes possible to consider the consistent extension of a consistent set of closed sequents.

Lemma 2.5.11. Let $\mathcal{C}$ be a structural sequent calculus with rules endowed with persistent provisos. If S is a consistent set of closed sequents and $S \nvdash_{\mathcal{C}} \to v_1, \dots, v_m$ for closed assertions $v_1, \dots, v_m$ then the set $S \cup \{\to \overline{v_1}, \dots, \to \overline{v_m}\}$ is still consistent.

Proof. Assume $S \cup \{\to \overline{v_1}, \dots, \to \overline{v_m}\}$ is inconsistent. So there is δ, closed assertion, such that $S, \to \overline{v_1}, \dots, \to \overline{v_m} \vdash_{\mathcal{C}} \to \delta$ and $S, \to \overline{v_1}, \dots, \to \overline{v_m} \vdash_{\mathcal{C}} \to \overline{\delta}$. By Theorem 2.3.2, $S, \to \overline{v_1}, \dots, \to \overline{v_m} \vdash_{\mathcal{C}} \to v_1$, using Theorem 2.3.3 $S \vdash_{\mathcal{C}} \overline{v_1}, \dots, \overline{v_m} \to v_1$ and applying Theorem 2.3.1 $S \vdash_{\mathcal{C}} \to v_1, v_1, \dots, v_m$. By right contraction $S \vdash_{\mathcal{C}} \to v_1, \dots, v_m$, contradicting the hypothesis that $S \nvdash_{\mathcal{C}} \to v_1, \dots, v_m$. Therefore, $S \cup \{\to \overline{v_1}, \dots, \to \overline{v_m}\}$ is consistent. □

Theorem 2.5.12. Every full structural sequent logic system with rules endowed with persistent provisos is complete.

Proof. Consider the logic $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ and let $\mathcal{C} = \langle \Sigma, \mathcal{R} \rangle$. Assume that $S \not\vdash_{\mathcal{C}} \Delta' \to \Delta''$ with $S \cup \{\Delta' \to \Delta''\}$ composed of closed sequents.

Given an enumeration v_n with $n \in \mathbb{N}$ of the set of closed assertions (notice that this set is enumerable since the components of a signature are enumerable sets), start by extending S to a maximal consistent set $S^\bullet$ as follows:

- $S_0 = S \cup \{\to \delta : \delta \in \overline{\Delta''} \cup \Delta'\}$;
- $S_{n+1} = \begin{cases} S \cup \{\to v_n\} & \text{provided that } S_n \vdash_{\mathcal{C}} \to v_n \\ S \cup \{\to \overline{v_n}\} & \text{otherwise} \end{cases}$;
- $S^\bullet = \cup_{n \in \mathbb{N}} S_n$.

Observe that $S^\bullet$ is still consistent thanks to Lemma 2.5.11. Furthermore, by construction, it is maximal consistent. Therefore, $S^\bullet \not\vdash_{\mathcal{C}} \Delta' \to \Delta''$ because otherwise $S^\bullet \vdash_{\mathcal{C}} \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$ and, hence, $S^\bullet$ would be inconsistent. Thus, by Lemma 2.5.9 applied to a closed sequent, $\mathbb{A}(\mathcal{C}, S^\bullet) \not\Vdash \Delta' \to \Delta''$.

On the other hand, for every $s \in S$, $S \vdash_{\mathcal{C}} s$ holds, and thus, again thanks to Lemma 2.5.9, $\mathbb{A}(\mathcal{C}, S^\bullet) \Vdash s$.

Since the logic is full and taking into account Lemma 2.5.10, $\mathbb{A}(\mathcal{C}, S^\bullet)$ is in $\mathcal{A}$. Hence, $S \not\vDash_{\mathcal{A}} \Delta' \to \Delta''$. □

Chapter 3

Modal Logics Labelled with Truth Values

Modal logics [BdRV01] are probably one of the most well known non-classical logics studied nowadays. They are presented by Kripke semantics, modal algebras, Hilbert systems, Gentzen systems, natural deduction and in many other ways. They have been applied in a wide list of areas like philosophy, mathematics, computer science, economy, etc. However most of the development and research of the field is concentrated in a fragment of the class of modal systems, namely the normal modal systems. In fact, usually the term "normal" is dropped and "modal logics" is quite often synonym of normal modal logics.

In this chapter a wide class of modal logics are presented as LSLTV: the normal modal logics are in Section 3.1 since they were already defined in [MSSV04, RSSV02b, RRS10]. The development of non-normal modal systems as LSLTV is new and is presented in Section 3.2. It is interesting to notice that accommodating non-normal modal calculi and semantics in this framework did not involve significant modifications.

3.1 (Normal) Modal Logics

This section presents (normal) modal logics as systems labelled with truth values. First the signature and the rules of the calculus are defined and illustrated with an example of a derivation. After that some remarks are made about the d-MTD and d-MTMP properties (defined in 2.3.4) for modal calculus. Then it is showed how a Σ_M-algebra is induced by a general Kripke structure and the other way around.

For illustration purposes, several well known (normal) modal logic systems are then presented.

Definition 3.1.1. A *signature* $\Sigma_M = \langle C, O, \mathbb{P}, X, Y, Z \rangle$ *for a modal logic with a modality* is such that:

- $C_0 = \{\mathbf{t}, \mathbf{f}\} \cup \{\mathbf{p}_i : i \in \mathbb{N}\}$;
- $C_1 = \{\neg, \Box\}$;
- $C_2 = \{\wedge, \vee, \supset\}$;
- $C_k = \varnothing$ for $k \geq 3$;
- $O_0 = \{\top, \bot\}$;
- $O_1 = \{\mathbf{I}, \mathbf{N}\}$;
- $O_2 = \{\mathbf{lb}\}$;
- $O_k = \varnothing$ for $k \geq 3$;
- $\mathbb{P}_1 = \{\Omega\}$;
- $\mathbb{P}_2 = \{\sqsubseteq\}$;
- $\mathbb{P}_k = \varnothing$ for $k \geq 3$;
- X, Y, Z are disjoint countable sets of variables. ∎

The formula constructors are the usual ones in modal logic. The operators are used namely to reason about the accessibility relation and to relate terms. In the case of the signature presented, $\mathbf{N}$ is the neighbourhood operator representing the collection of neighbours according to the accessibility relation associated to the denotation of a term, that is, if terms are seen as set of worlds, $\mathbf{N}(\tau_1)$ is the set of all worlds accessible from at least one of the worlds in τ_1.

For (normal) modal logics several kinds of rules are needed: rules for connectives, rules for operators, structural rules, etc. See Table 3.1 for the specific rules for the modal constructor and the operator $\mathbf{N}$.

Observe the intrinsic relation between the $\mathbf{N}$ operator and the modality. Meanwhile, the rules for $\mathbf{N}$ state that the neighbourhood of a truth value is induced by the neighbourhoods of the atomic truth values contained in it.

$$\text{L}\Box \; \frac{\mathbf{N}(\tau_1) \leq \xi_1, \Gamma_1 \to \Gamma_2}{\tau_1 \leq (\Box\xi_1), \Gamma_1 \to \Gamma_2} \qquad \text{R}\Box \; \frac{\Gamma_1 \to \Gamma_2, \mathbf{N}(\tau_1) \leq \xi_1}{\Gamma_1 \to \Gamma_2, \tau_1 \leq (\Box\xi_1)}$$

$$\text{L}\mathbf{N}\Omega \; \frac{\tau_3 \sqsubseteq \tau_1, \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \not\sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \not\sqsubseteq \mathbf{N}(\tau_1)} \lhd \tau_3 : \mathbf{y}, \tau_3 \notin \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

$$\text{R}\mathbf{N}\Omega \; \frac{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3 \quad \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_3 \sqsubseteq \tau_1 \quad \Omega\tau_3, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_3)}{\Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}(\tau_1)}$$

Table 3.1: Specific rules for the modal constructor and operator

Definition 3.1.2. A *sequent calculus for a normal modal logic* with a modality is the pair $\mathcal{C}_M = \langle \Sigma_M, \mathcal{R}_M \rangle$ where Σ_M is the signature in Definition 3.1.1 and $\mathcal{R}_M$ is the set containing the rules in Tables 2.3, 3.1, 2.4, 2.2 and 2.1 (where the rules showed with ϱ are presented in Table 3.2). ■

It is now presented an example of a derivation in the context of a sequent calculus for a normal modal logic.

Example 3.1.3. In the context of the sequent calculus introduced in Definition 3.1.2, the following holds: $\Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \mathbf{y}_1 \leq \psi_2 \to \mathbf{y}_1 \leq \varphi \vdash_{\mathcal{R}} \to \top \leq (\psi_1 \wedge \psi_2) \supset \varphi$.

1.		$\to$	$\top \leq (\psi_1 \wedge \psi_2) \supset \varphi$	RgenF 2
2.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top$	$\to$	$\mathbf{y}_1 \leq (\psi_1 \wedge \psi_2) \supset \varphi$	R$\supset$ 3
3.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top$, $\mathbf{y}_1 \leq \psi_1 \wedge \psi_2$	$\to$	$\mathbf{y}_1 \leq \varphi$	L$\wedge$ 4
4.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top$, $\mathbf{y}_1 \leq \psi_1, \mathbf{y}_1 \leq \psi_2$	$\to$	$\mathbf{y}_1 \leq \varphi$	Lw 5
5.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \leq \psi_1, \mathbf{y}_1 \leq \psi_2$	$\to$	$\mathbf{y}_1 \leq \varphi$	hyp ■

Example 3.1.4. Each sequent calculus for (normal) modal logic has l-MTD, l-MTMP and g-MTMP, all with $\Lambda = \{\tau_1 \leq \xi_1 \supset \xi_2\}$.

The proofs are the same as in Example 2.3.6 in classical propositional logic. It is worthwhile to notice that the proof of g-MTD in that example is not sound in the context of (normal) modal logic since the rule **1** does not belong to $\mathcal{R}_M$. ■

One of the most common ways to express the semantics of modal logics is by Kripke structures. Among Kripke structures, general Kripke structures deserve a particular importance since they provide a complete semantics to some modal logics not complete with respect to the standard Kripke semantics. For instance, the system $\mathcal{GL}$, the normal modal logic generated by the Löb axiom

$$\Box(\Box\varphi \supset \varphi) \supset \Box\varphi$$

AxT	$\overline{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$	AxΩ	$\overline{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \Omega\tau_1}$
LwT	$\dfrac{\Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RwT	$\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LwΩ	$\dfrac{\Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2}$	RwΩ	$\dfrac{\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$
LcT	$\dfrac{\tau_1 \sqsubseteq \tau_2, \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RcT	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2, \tau_1 \sqsubseteq \tau_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LcΩ	$\dfrac{\Omega\tau_1, \Omega\tau_1, \Gamma_1 \to \Gamma_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2}$	RcΩ	$\dfrac{\Gamma_1 \to \Gamma_2, \Omega\tau_1, \Omega\tau_1}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$
LxiT	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RxiT	$\dfrac{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}$
LxiΩ	$\dfrac{\Gamma_1 \to \Gamma_2, \Omega\tau_1}{\mho\tau_1, \Gamma_1 \to \Gamma_2}$	RxiΩ	$\dfrac{\Omega\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \mho\tau_1}$
LxeT	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2}{\tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}$	RxeT	$\dfrac{\tau_1 \not\sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2}$
LxeΩ	$\dfrac{\Gamma_1 \to \Gamma_2, \mho\tau_1}{\Omega\tau_1, \Gamma_1 \to \Gamma_2}$	RxeΩ	$\dfrac{\mho\tau_1, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2, \Omega\tau_1}$
cutT	$\dfrac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \tau_1 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$	cutΩ	$\dfrac{\Gamma_1 \to \Gamma_2, \Omega\tau_1 \quad \Omega\tau_1\Gamma_1 \to \Gamma_2}{\Gamma_1 \to \Gamma_2}$

Table 3.2: Structural rules for $\sqsubseteq$

is not strongly complete with respect to standard Kripke structures (see [BdRV01, p. 211]). A general Kripke structure is a tuple $K = \langle W, \mathcal{B}, \rightsquigarrow, V\rangle$ where W is the set of possible worlds, $\mathcal{B} \subseteq 2^W$ is the set of admissible valuations, $\rightsquigarrow \subseteq W^2$ is the accessible relation for the modal constructor $\Box$ and $V : \Pi \to \mathcal{B}$ is the valuation map. These general Kripke structures can be induced as Σ_M-algebras as showed in the sequel.

Definition 3.1.5. From any general Kripke structure $K = \langle W, \mathcal{B}, \rightsquigarrow, V\rangle$, it can be defined an algebra $\mathbb{A}_K = \langle F, T, \cdot_{\mathbb{A}_K}\rangle$ where:

- $F = \mathcal{B}$;
- $T = 2^W$;
- $\#_{\mathbb{A}_K} = \lambda b.b$;
- $a \in \Omega_{\mathbb{A}_K}$ iff a is a singleton;
- $\langle a, a'\rangle \in \sqsubseteq_{\mathbb{A}_K}$ iff $a \subseteq a'$;
- $\langle a, b\rangle \in \leq_{\mathbb{A}_K}$ iff $a \subseteq b$;

- $\bot_{\mathbb{A}_K} = \varnothing$;
- $\top_{\mathbb{A}_K} = W$;
- $\mathbf{I}_{\mathbb{A}_K} = \lambda a.\iota(a)$;
- $\mathbf{N}_{\mathbb{A}_K} = \lambda a.\{w' \in W : \text{exists } w \in a \text{ such that } w \rightsquigarrow w'\}$;
- $\mathbf{lb}_{\mathbb{A}_K} = \lambda aa'.a \cap a'$;
- $\mathbf{f}_{\mathbb{A}_K} = \varnothing$;
- $\mathbf{t}_{\mathbb{A}_K} = W$;
- $\mathbf{p}_{i\mathbb{A}_K} = V(\mathbf{p}_i)$;
- $\neg_{\mathbb{A}_K} = \lambda b.W \setminus b$;
- $\Box_{\mathbb{A}_K} = \lambda b.\{w \in W : \mathbf{N}_{\mathbb{A}_K}(\{w\}) \subseteq b\}$;
- $\wedge_{\mathbb{A}_K} = \lambda bb'.b \cap b'$;
- $\vee_{\mathbb{A}_K} = \lambda bb'.b \cup b'$;
- $\supset_{\mathbb{A}_K} = \lambda bb'.(W \setminus b) \cup b'$;

with ι being a choice function for W. ■

Definition 3.1.6. From a Σ_M-algebra $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}}\rangle$ appropriate for $\mathcal{C}_M$, it can be defined a general Kripke structure $K_{\mathbb{A}} = \langle W, \rightsquigarrow, \mathcal{B}, V\rangle$ where:

- $W = \Omega_{\mathbb{A}}$;
- $t \rightsquigarrow t'$ iff $t' \sqsubseteq_{\mathbb{A}} \mathbf{N}_{\mathbb{A}}(t)$;
- $\mathcal{B} = \{\langle f\rangle_{\mathbb{A}} : f \in F\}$ with $\langle f\rangle_{\mathbb{A}} = \{t \in \Omega_{\mathbb{A}} : t \leq_{\mathbb{A}} f\}$;
- $V(\mathbf{p}_i) = \langle \mathbf{p}_{i\mathbb{A}}\rangle_{\mathbb{A}}$. ■

Entailment is preserved (both local and global, as in Definition 2.4.8) between a general Kripke structure K and its induced algebra $\mathbb{A}_K$, that is,

$$\psi_1, \ldots, \psi_k \vDash^d_K \varphi \text{ iff } \psi_1, \ldots, \psi_k \vDash^d_{\mathbb{A}_K} \varphi \text{ for } d = l, g,$$

as well between an algebra $\mathbb{A}$ and its induced general Kripke structure $K_{\mathbb{A}}$, that is,

$$\psi_1, \ldots, \psi_k \vDash^d_{\mathbb{A}} \varphi \text{ iff } \psi_1, \ldots, \psi_k \vDash^d_{K_{\mathbb{A}}} \varphi \text{ for } d = l, g,$$

where $\vDash_K$ and $\vDash_{K_{\mathbb{A}}}$ are the usual entailment relation for general Kripke structures. Both proofs can be found in section 3.3 of [MSSV04] as well as the proof that for every general Kripke structure K the induced Σ_M-algebra $\mathbb{A}_K$ is appropriate for each rule in $\mathcal{R}_M$.

The next example shows that the axiom for reflexivity is satisfied by the Σ_M-algebra induced by a general Kripke structure that is *not* reflexive. Actually, the axiom characterizes reflexivity only among standard frames. But it is possible to characterize general frames with a reflexive accessibility relation using rule T

$$\frac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}(\tau_1)} \ .$$

Example 3.1.7. Consider the general Kripke structure $\langle W, \mathcal{B}, \leadsto, V\rangle$ where $W = \{w_1, w_2\}$, $\leadsto = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle\}$, $\mathcal{B} = \{\varnothing, W\}$ and V such that $\mathbf{p}_1 = \varnothing$ and $\mathbf{p}_2 = W$. Let $\mathbb{A}$ be the Σ_M-algebra induced by this general Kripke structure according to Example 3.1.5. Thus, $\mathbb{A} \Vdash \top \leq (\Box \mathbf{p}_i) \supset \mathbf{p}_i$ for $i = 1, 2$ as showed below. In fact:

- $[\![\top]\!]_{\mathbb{A}} = W$;
- $[\![\mathbf{p}_1]\!]_{\mathbb{A}} = \varnothing$;
- $[\![\mathbf{p}_2]\!]_{\mathbb{A}} = W$;
- $\Box_{\mathbb{A}}(\varnothing) = \varnothing$;
- $\Box_{\mathbb{A}}(W) = W$.

Then

$$\begin{array}{lll} \mathbb{A} \Vdash \top \leq (\Box \mathbf{p}_1) \supset \mathbf{p}_1 & \text{iff} & \langle [\![\top]\!]_{\mathbb{A}}, [\![(\Box \mathbf{p}_1) \supset \mathbf{p}_1]\!]_{\mathbb{A}}\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}([\![\Box \mathbf{p}_1]\!]_{\mathbb{A}}, [\![\mathbf{p}_1]\!]_{\mathbb{A}})\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}(\Box_{\mathbb{A}}([\![\mathbf{p}_1]\!]_{\mathbb{A}}), [\![\mathbf{p}_1]\!]_{\mathbb{A}})\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}(\Box_{\mathbb{A}}(\varnothing), \varnothing)\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}(\varnothing, \varnothing)\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, W\rangle \in \leq_{\mathbb{A}} \ . \end{array}$$

And, for $\mathbf{p}_2$

$$\begin{array}{lll} \mathbb{A} \Vdash \top \leq (\Box \mathbf{p}_2) \supset \mathbf{p}_2 & \text{iff} & \langle [\![\top]\!]_{\mathbb{A}}, [\![(\Box \mathbf{p}_2) \supset \mathbf{p}_2]\!]_{\mathbb{A}}\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}([\![\Box \mathbf{p}_2]\!]_{\mathbb{A}}, [\![\mathbf{p}_2]\!]_{\mathbb{A}})\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}(\Box_{\mathbb{A}}([\![\mathbf{p}_2]\!]_{\mathbb{A}}), [\![\mathbf{p}_2]\!]_{\mathbb{A}})\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}(\Box_{\mathbb{A}}(W), W)\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, \supset_{\mathbb{A}}(W, W)\rangle \in \leq_{\mathbb{A}} \\ & \text{iff} & \langle W, W\rangle \in \leq_{\mathbb{A}} \end{array}$$

∎

It is now presented the definition of modal logic system. When the set of rules is only the one included in Definition 3.1.2, the system characterized is the smaller normal modal system usually called K.

Definition 3.1.8. A *(normal) modal logic system* $\mathcal{L}_M$ is a relational logic system $\langle \Sigma_M, \mathcal{R}_M, \mathcal{A} \rangle$ where the signature Σ_M is as Definition 3.1.1, the rules are at least those described in Definition 3.1.2 and $\mathcal{A}$ is a class of Σ_M-algebras as in Definition 2.4.1. ■

Observe that in a modal logic system the following consequences hold:

- $\varphi \vdash^g_{\mathcal{R}} \Box\varphi$ and
- $\vdash^g_{\mathcal{R}} \Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$.

As practical examples of modal logic systems, a deontic logic system, a Löb provability logic system and a knowledge logic system are presented below.

Example 3.1.9. Consider the modal logic system $\mathsf{SDL} = \langle \Sigma_D, \mathcal{R}_D, \mathcal{A}_D \rangle$ for the Standard Deontic Logic, see [Hil02, section 6], where:

- Σ_D is a modal signature as in Definition 3.1.1 where $\Box\varphi$ is intended to mean that φ is obligatory;
- $\mathcal{R}_D$ is the set of rules as in Definition 3.1.2 plus the rule D:

$$\frac{}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \mathbf{N}(\tau_1) \not\sqsubseteq \bot}$$

  expressing that for any atomic truth value there is at least another truth value *acceptable* by it, i.e., everything obligatory associated to the atomic truth value holds in that other truth value;

- $\mathcal{A}_D$ is the class of all Σ_D-algebras appropriate for $\langle \Sigma_D, \mathcal{R}_D \rangle$. It worthwhile to notice that $\mathcal{A}_D$ includes the Σ_D-algebras induced by general Kripke structures where the accessible relation is right-unbounded. This is guaranteed by a characterization theorem given in [MSSV04, p.257–258]. ■

Deontic logic systems are concerned with what is obligatory at a certain point. It has been widely used in a multitude of fields ranging from computer science to philosophy or even in law. However the definition of deontic systems as a normal modal logic has some side-effects, as theorems that belongs to normal modal logics that should be relaxed for the deontic reasoning. For instance, if the system has

$\neg\Box\bot$ as an axiom with the intended meaning that there isn't impossible obligations, then in a normal system the statement $\neg(\Box\varphi \wedge \Box\neg\varphi)$, with the intended meaning that there is no conflict of obligations, always holds. Nevertheless the second formula should be relaxed if the existence of moral dilemmas in the system could occur. This can be solved using non-normal modal logics as the ones in Section 3.2.

It worthwhile to remember the concept of *conversely well-founded relation* for the following example. A relation is conversely well-founded if and only if there no infinite ascending sequences.

The logic system $\mathcal{GL}$ is concerned with reasoning about provability, more specifically about what can be expressed by arithmetical theories about their provability predicates, and was considered for instance, when dealing with the logical omniscience problem, and in dealing with reflection in artificial intelligence, automated deduction and verification.

Example 3.1.10. Consider the modal logic system $\mathcal{GL} = \langle \Sigma_{GL}, \mathcal{R}_{GL}, \mathcal{A}_{GL} \rangle$ corresponding to the modal Löb provability logic for a certain theory **T**, see [AB04], where:

- Σ_{GL} is a modal signature as in Definition 3.1.1 where $\Box\varphi$ is intended to mean that φ is provable in **T**;
- $\mathcal{R}_{GL}$ is the set of rules described in Definition 3.1.2 plus the rule W:

$$\frac{\Omega\tau_1, \Omega\tau_3, \tau_3 \sqsubseteq \mathbf{N}(\tau_1), \Gamma_1 \to \Gamma_2, \tau_3 \sqsubseteq \tau_2, \mathbf{N}(\tau_3) \not\sqsubseteq \tau_2}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \mathbf{N}(\tau_1) \sqsubseteq \tau_2} \triangleleft \tau_3 : \mathbf{y}, \tau_3 \notin \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

- $\mathcal{A}_{GL}$ is the class of all Σ_{GL}-algebras appropriate for $\langle \Sigma_{GL}, \mathcal{R}_{GL} \rangle$. It is worthwhile to notice that this includes the Σ_{GL}-algebras induced by general Kripke structures where the accessibility relation is transitive and conversely well-founded (by the characterization theorem in [MSSV04, p.257–258]). ■

A multi-modal logic system, for a knowledge logic system is now presented. This system can actually be viewed as an example of fusion of n modal logic systems, as it will be clear when fusion is introduced later on.

Example 3.1.11. Consider a knowledge logic system $\mathcal{K}$ for n agents where each modal formula $K_i\varphi$ is intended to mean that agent i knows φ, see [Hal95]. That is, $\mathcal{K} = \langle \Sigma_K, \mathcal{R}_K, \mathcal{A}_K \rangle$ is such that:

- Σ_K is a modal signature as described in Definition 3.1.1 with $C_1 = \{\neg, K_1, \ldots, K_n\}$ and $O_1 = \{\mathbf{I}, \mathbf{N}_1, \ldots, \mathbf{N}_n\}$;

- $\mathcal{R}_K$ is the set of rules from Definition 3.1.2 with n copies of the rules L$\Box$, R$\Box$, L$\mathbf{N}\Omega$ and R$\mathbf{N}\Omega$ renamed as LK_i, RK_i, L$\mathbf{N}_i\Omega$ and R$\mathbf{N}_i\Omega$ for $i = 1, \ldots, n$, together with the rules listed in Table 3.3. As it is clear from the inspection of the rules, T_i imposes that each associated accessibility relation is reflexive, 4_i that it is transitive, and the rule 5_i that each accessibility relation is Euclidean. Together they impose that each relation is an equivalence;
- $\mathcal{A}_K$ is the class of all Σ_K-algebras appropriate for $\langle \Sigma_K, \mathcal{R}_K \rangle$. Observe that this class includes the Σ_K-algebras induced by general Kripke structures where there are n accessibility relations which are equivalence relations (see [MSSV04, p.257–258]). ■

$$\frac{}{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}_i(\tau_1)}\ \mathrm{T}_i \qquad \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{N}_i(\mathbf{N}_i(\tau_1)) \sqsubseteq \mathbf{N}_i(\tau_1)}\ 4_i$$

$$\frac{\Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}_i(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Omega\tau_3, \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \mathbf{N}_i(\tau_3) \quad \Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \Omega\tau_3}{\Omega\tau_1, \Omega\tau_2, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{N}_i(\tau_2)}\ 5_i$$

Table 3.3: Additional rules for the Knowledge Logic System $\mathcal{K}$

3.2 Non-normal Modal Logics

Non-normal modal logics are an extension of propositional logics with a modal constructor $\Box$. The calculus usually does not include the necessitation rule (from φ infer $\Box\varphi$) and/or the K axiom $\Box(\varphi_1 \supset \varphi_2) \supset (\Box\varphi_1 \supset \Box\varphi_2)$. The semantics is a variant of Kripke semantics usually called in the literature neighbourhood semantics.

Despite that the invention of neighbourhood semantics and the minimal models semmantics started in the 1970's (independent publications [Sco70] and [Mon70]), they seem not be so used in comparison with applications as normal ones. Still some properties were investigated by Segerberg [Seg71] and Chellas [Che80], for instance. However nowadays it is possible to find non-normal modal systems for a little bunch of areas like law [Jon90], game theory [Par85], coalition logics [Pau02], concurrent propositional dynamic logic [Gol92], and belief modalities [HL00] for instance.

The framework of LSLTV can be easily used to deal with non-normal modal logics and, as presented in this section, the whole hierarchy presented in [Che80] of non-normal modal systems can be expressed in this context. With this kind of logics represented as LSLTV and taking into account the definition of combination in Chapter 4, new results about fusion of non-normal modal logics can be obtained.

For a gentle introduction to non-normal modal systems, see [Che80].

Definition 3.2.1. A structure $\mathcal{S}$ for a non-normal modal logic is a triple $\langle W, f, V\rangle$ where:

- W is a set of worlds,
- $f : 2^W \to 2^W$ is a function, and
- $V : \{\mathbf{p}_i : i \in \mathbb{N}\} \to 2^W$ is a valuation map.

The *denotation* of a formula ψ in $\mathcal{S}$ is inductively defined as follows:

- $[\![\psi]\!]^{\mathcal{S}} = V(\psi)$, if ψ is a propositional symbol,
- $[\![\neg\,\psi]\!]^{\mathcal{S}} = W \setminus [\![\psi]\!]^{\mathcal{S}}$,
- $[\![\psi \wedge \psi']\!]^{\mathcal{S}} = [\![\psi]\!]^{\mathcal{S}} \cap [\![\psi']\!]^{\mathcal{S}}$,
- $[\![\psi \supset \psi']\!]^{\mathcal{S}} = [\![\psi']\!]^{\mathcal{S}} \cup W \setminus [\![\psi]\!]^{\mathcal{S}}$,
- $[\![\Box\psi]\!]^{\mathcal{S}} = f([\![\psi]\!]^{\mathcal{S}})$. ■

In order to represent structures for non-normal modal logics in the LSLTV approach, it is convenient to assume that they use a function $f : 2^W \to 2^W$ instead of either the addition of a set $Q \subseteq W$ of queer worlds in a Kripke structure as proposed by [Kri65], or the operator $N : W \to 2^{2^W}$ as the minimal models in [Che80]. In fact, this formulation with the function f is pointwise equivalent with minimal models (see [Che80, p. 211]).

The denotation of the non-modal fragment is as in normal modal logics. For the modality $\Box\varphi$ the denotation is directly the application of the funtion f to the denotation of the formula φ. It is worthwhile to notice that no restrictions are made about f.

A signature for non-normal modal logic in the context of LSLTV is as follows.

Definition 3.2.2. A *signature* $\Sigma_N = \langle C, O, \sqsubseteq, X, Y, Z\rangle$ *for a non-normal modal logic with a modality* is such that:

- $C_0 = \{\mathbf{t}, \mathbf{f}\} \cup \{\mathbf{p}_i : i \in \mathbb{N}\}$;
- $C_1 = \{\neg, \Box\}$;
- $C_2 = \{\wedge, \vee, \supset\}$;

- $C_k = \varnothing$ for $k \geq 3$;
- $O_0 = \{\top, \bot\}$;
- $O_1 = \{\mathbf{I}, \mathbf{F}\}$;
- $O_2 = \{\mathbf{lb}\}$;
- $O_k = \varnothing$ for $k \geq 3$;
- $\mathsf{P}_1 = \{\Omega\}$;
- $\mathsf{P}_2 = \{\sqsubseteq\}$;
- $\mathsf{P}_k = \varnothing$ for $k \geq 3$;
- X, Y, Z are disjoint countable sets of variables. ■

The only difference between the signature of a non-normal and a normal modal logic is the operator **F** instead of **N**. This new operator will play the role of the function f in Definition 3.2.1.

The rules for the modal constructor and the **F** operator are presented in Table 3.4 and are different from the corresponding ones for normal modal logics. It can be seen right away the relation between the rules for $\Box$ and the last item in Definition 3.2.1. The rule eq**F** states the only condition about the function f that must be reflected in the **F** operator: for equal terms, the result of **F** must be the same.

$$\text{L}\Box\ \frac{\tau_1 \sqsubseteq \mathbf{F}(\#\xi_1), \Gamma_1 \to \Gamma_2}{\tau_1 \leq \Box\xi_1, \Gamma_1 \to \Gamma_2} \qquad \text{R}\Box\ \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{F}(\#\xi_1)}{\Gamma_1 \to \Gamma_2, \tau_1 \leq \Box\xi_1}$$

$$\text{eq}\mathbf{F}\ \frac{\Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Gamma_1 \to \Gamma_2, \tau_2 \sqsubseteq \tau_1}{\Gamma_1 \to \Gamma_2, \mathbf{F}(\tau_1) \sqsubseteq \mathbf{F}(\tau_2)}$$

$$\mathbf{I}\ \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{I}(\tau_1) \sqsubseteq \tau_1} \qquad \Omega\mathbf{I}\ \frac{\Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \bot}{\Gamma_1 \to \Gamma_2, \Omega\mathbf{I}(\tau_1)}$$

$$\mathbf{lb}1\ \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{lb}(\tau_1, \tau_2) \sqsubseteq \tau_1} \qquad \mathbf{lb}2\ \frac{}{\Gamma_1 \to \Gamma_2, \mathbf{lb}(\tau_1, \tau_2) \sqsubseteq \tau_2}$$

Table 3.4: Rules for the modal constructor and the operators in non-normal calculi

Definition 3.2.3. A *sequent calculus for a non-normal modal logic* with a modality is the pair $\mathcal{C}_N = \langle \Sigma_N, \mathcal{R}_N \rangle$ where Σ_N is the signature in Definition 3.2.2 and $\mathcal{R}_N$ is the set constituted at least by the rules in Tables 2.3, 3.4, 2.2 and 2.1 (where the rules showed with ϱ are presented in Table 3.2). ■

In the sequel it is proved that $\Box$ preserves equivalence theoremhood, i.e., it is presented a derivation for $\Box\varphi_1 \Leftrightarrow \Box\varphi_2$ assuming as hypothesis $\varphi_1 \Leftrightarrow \varphi_2$. Actually, this is the only property with respect to $\Box$ that is imposed in non-normal modal logics. For this purpose the rules for the connective $\Leftrightarrow$ presented in Table 3.5, obtained by seeing $\Leftrightarrow$ as the abbreviation $\xi_1 \Leftrightarrow \xi_2 \equiv (\xi_1 \supset \xi_2) \wedge (\xi_2 \supset \xi_1)$, are used.

$$\text{L}\Leftrightarrow \quad \frac{\Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1, \tau_1 \leq \xi_2 \quad \tau_1 \leq \xi_1, \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2}{\tau_1 \leq \xi_1 \Leftrightarrow \xi_2, \Gamma_1 \to \Gamma_2}$$

$$\text{R}\Leftrightarrow \quad \frac{\Omega\tau_1, \tau_1 \leq \xi_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_2 \quad \Omega\tau_1, \tau_1 \leq \xi_2, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \leq \xi_1 \Leftrightarrow \xi_2}$$

Table 3.5: Rules for the equivalence connective

Example 3.2.4. In the context of a sequent calculus for non-normal modal logics as in Definition 3.2.3, $\to \top \leq \varphi_1 \Leftrightarrow \varphi_2 \vdash_{\mathcal{C}_N} \to \top \leq \Box\varphi_1 \Leftrightarrow \Box\varphi_2$.

1.		$\to$	$\top \leq \Box\varphi_1 \Leftrightarrow \Box\varphi_2$	RgenF 2
2.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top$	$\to$	$\mathbf{y_1} \leq \Box\varphi_1 \Leftrightarrow \Box\varphi_2$	R$\Leftrightarrow$ 3,3'
3.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \leq \Box\varphi_1$	$\to$	$\mathbf{y_1} \leq \Box\varphi_2$	R$\Box$ 4
3'.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \leq \Box\varphi_2$	$\to$	$\mathbf{y_1} \leq \Box\varphi_1$	see note (B)
4.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \leq \Box\varphi_1$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	L$\Box$ 5
5.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	transT 6,7
6.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	axiom
7.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\to$	$\mathbf{F}(\#\varphi_1) \sqsubseteq \mathbf{F}(\#\varphi_2)$	Lw's 8
8.		$\to$	$\mathbf{F}(\#\varphi_1) \sqsubseteq \mathbf{F}(\#\varphi_2)$	eq**F** 9,9'
9.		$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2$	cutF 10,11
9'.		$\to$	$\#\varphi_2 \sqsubseteq \#\varphi_1$	see note (A)
10.		$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2, \#\varphi_1 \leq \varphi_1 \Leftrightarrow \varphi_2$	transF 12,13
11.	$\#\varphi_1 \leq \varphi_1 \Leftrightarrow \varphi_2$	$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2$	L$\Leftrightarrow$ 14,15
12.		$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2, \#\varphi_1 \sqsubseteq \top$	$\top$
13.		$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2, \top \leq \varphi_1 \Leftrightarrow \varphi_2$	hyp
14.		$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2, \#\varphi_1 \leq \varphi_1,$ $\#\varphi_1 \leq \varphi_2$	cutT 16,17
15.	$\#\varphi_1 \leq \varphi_1, \#\varphi_1 \leq \varphi_2$	$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2$	R# 19
16.		$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2, \#\varphi_1 \leq \varphi_1,$ $\#\varphi_1 \leq \varphi_2, \#\varphi_1 \sqsubseteq \#\varphi_1$	ref
17.	$\#\varphi_1 \sqsubseteq \#\varphi_1$	$\to$	$\#\varphi_1 \sqsubseteq \#\varphi_2, \#\varphi_1 \leq \varphi_1,$ $\#\varphi_1 \leq \varphi_2$	L# 18

18.	$\#\varphi_1 \leq \varphi_1$	$\rightarrow$	$\#\varphi_1 \sqsubseteq \#\varphi_2, \#\varphi_1 \leq \varphi_1,$ $\#\varphi_1 \leq \varphi_2$	axiom
19.	$\#\varphi_1 \leq \varphi_1, \#\varphi_1 \leq \varphi_2$	$\rightarrow$	$\#\varphi_1 \leq \varphi_2$	axiom

Notes:

(A) The proof of line 9' is given by lines 10 to 19 changing all the occurrences of φ_1 by φ_2 and vice-versa.

(B) The proof of line 3' is given by lines 4 to 19 changing all the occurrences of φ_2 by φ_1 and vice-versa. ■

Similarly to the concept of general Kripke structure, there are general structures for non-normal modal logic. As in the normal case, there are systems that are complete only with these kind of structures [Ger75]. About this subject, see [Han03, KW99].

Definition 3.2.5. A *general structure $\mathcal{S}$ for a non-normal modal logic* is $\langle W, f, A, V\rangle$ where:

- W is a set of worlds;
- $f : 2^W \rightarrow 2^W$ is a function;
- $A \subseteq 2^W$ is such that
  - $\varnothing \in A$,
  - if $X, Y \in A$ then $X \cup Y \in A$,
  - if $X \in A$ then $W \setminus X \in A$,
  - if $X \in A$ then $f(X) \in A$;
- $V : \{\mathbf{p}_i : i \in \mathbb{N}\} \rightarrow A$ is a valuation map. ■

It is possible to induce a Σ_N-algebra from a general structure for a non-normal logic presented in Definition 3.2.5. Moreover global and local entailment are equivalent in both semantics.

Definition 3.2.6. Given a general structure $\mathcal{S} = \langle W, f, A, V\rangle$ of non-normal modal logic the corresponding Σ_N-algebra $\mathbb{A}_\mathcal{S} = \langle F, T, \cdot_{\mathbb{A}_\mathcal{S}}\rangle$ is such that:

- *$F = A$;*
- *$T = 2^W$;*

- $\#_{\mathbb{A}_\mathcal{S}} = \lambda a.a$;
- $x \in \Omega_{\mathbb{A}_\mathcal{S}}$ iff x is a singleton;
- $\langle x, x' \rangle \in \sqsubseteq_{\mathbb{A}_\mathcal{S}}$ iff $x \subseteq x'$;
- $\langle x, a \rangle \in \leq_{\mathbb{A}_\mathcal{S}}$ iff $x \subseteq a$;
- $\bot_{\mathbb{A}_\mathcal{S}} = \varnothing$;
- $\top_{\mathbb{A}_\mathcal{S}} = W$;
- $\mathbf{I}_{\mathbb{A}_\mathcal{S}} = \lambda x.\iota(x)$;
- $\mathbf{lb}_{\mathbb{A}_\mathcal{S}} = \lambda x x'.x \cap x'$;
- $\mathbf{F}_{\mathbb{A}_\mathcal{S}} = \lambda x.f(x)$;
- $\mathbf{f}_{\mathbb{A}_\mathcal{S}} = \varnothing$;
- $\mathbf{t}_{\mathbb{A}_\mathcal{S}} = W$;
- $\mathbf{p_i}_{\mathbb{A}_\mathcal{S}} = V(\mathbf{p_i})$;
- $\neg_{\mathbb{A}_\mathcal{S}} = \lambda a.W \setminus a$;
- $\Box_{\mathbb{A}_\mathcal{S}} = \lambda a.\mathbf{F}_{\mathbb{A}_\mathcal{S}}(a)$;
- $\wedge_{\mathbb{A}_\mathcal{S}} = \lambda a a'.a \cap a'$;
- $\vee_{\mathbb{A}_\mathcal{S}} = \lambda a a'.a \cup a'$;
- $\supset_{\mathbb{A}_\mathcal{S}} = \lambda a a'.a' \cup W \setminus a$;

with ι being a choice function for W. ■

The following facts are straightforward to verify and, because of them, in the proofs of the following semantic results it is enough to prove statements about $\mathbf{f}, \mathbf{p}_i, \neg, \vee$ and $\Box$.

- $\mathbf{t}_{\mathbb{A}_\mathcal{S}} = \neg_{\mathbb{A}_\mathcal{S}}(\mathbf{f}_{\mathbb{A}_\mathcal{S}})$,
- $\supset_{\mathbb{A}_\mathcal{S}}(a, a') = \vee_{\mathbb{A}_\mathcal{S}}(\neg_{\mathbb{A}_\mathcal{S}}(a), a')$,
- $\wedge_{\mathbb{A}_\mathcal{S}}(a, a') = \neg_{\mathbb{A}_\mathcal{S}}(\vee_{\mathbb{A}_\mathcal{S}}(\neg_{\mathbb{A}_\mathcal{S}}(a), \neg_{\mathbb{A}_\mathcal{S}}(a')))$

It is showed now that the proposed entailment for non-normal modal logics is equivalent to the original one given in Definition 3.2.1.

Lemma 3.2.7. Let $\mathcal{S}$ be a general structure for a non-normal modal logic. Then, for every closed simple formula φ, $[\![\varphi]\!]^\mathcal{S} = [\![\varphi]\!]_{\mathbb{A}_\mathcal{S}}$.

Proof. The result follows by structural induction on φ.

- φ is $\mathbf{f}$. Then $[\![\mathbf{f}]\!]^{\mathcal{S}} = \varnothing = \mathbf{f}_{\mathbb{A}_{\mathcal{S}}} = [\![\mathbf{f}]\!]_{\mathbb{A}_{\mathcal{S}}}$.
- φ is $\mathbf{p}_i$. Then $[\![\mathbf{p}_i]\!]^{\mathcal{S}} = V(\mathbf{p}_i) = \mathbf{p}_{i_{\mathbb{A}_{\mathcal{S}}}} = [\![\mathbf{p}_i]\!]_{\mathbb{A}_{\mathcal{S}}}$.
- φ is $\neg\,\varphi'$. Then $[\![\neg\,\varphi']\!]^{\mathcal{S}} = W \setminus [\![\varphi']\!]^{\mathcal{S}} = W \setminus [\![\varphi']\!]_{\mathbb{A}_{\mathcal{S}}} = \neg_{\mathbb{A}_{\mathcal{S}}}([\![\varphi']\!]_{\mathbb{A}_{\mathcal{S}}}) = [\![\neg\,\varphi']\!]_{\mathbb{A}_{\mathcal{S}}}$.
- φ is $\Box\varphi'$. Then $[\![\Box\varphi']\!]^{\mathcal{S}} = f([\![\varphi']\!]^{\mathcal{S}}) = \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}([\![\varphi']\!]_{\mathbb{A}_{\mathcal{S}}}) = \Box_{\mathbb{A}_{\mathcal{S}}}([\![\varphi']\!]_{\mathbb{A}_{\mathcal{S}}}) = [\![\Box\varphi']\!]_{\mathbb{A}_{\mathcal{S}}}$.
- φ is $\varphi' \vee \varphi''$. Then $[\![\varphi' \vee \varphi'']\!]^{\mathcal{S}} = [\![\varphi']\!]^{\mathcal{S}} \cup [\![\varphi'']\!]^{\mathcal{S}} = [\![\varphi']\!]_{\mathbb{A}_{\mathcal{S}}} \cup [\![\varphi'']\!]_{\mathbb{A}_{\mathcal{S}}} = \vee_{\mathbb{A}_{\mathcal{S}}}([\![\varphi']\!]_{\mathbb{A}_{\mathcal{S}}}, [\![\varphi'']\!]_{\mathbb{A}_{\mathcal{S}}}) = [\![\varphi' \vee \varphi'']\!]_{\mathbb{A}_{\mathcal{S}}}$.

□

With this result it is possible to prove the satisfiability of a formula in a structure and in the correspondent algebra. This is showed in the next two propositions.

Proposition 3.2.8. Given a general structure $\mathcal{S}$ for non-normal modal logic and a closed simple formula φ, $\mathcal{S} \Vdash \varphi$ iff $\mathbb{A}_{\mathcal{S}} \Vdash \top \leq \varphi$.

Proof. $\mathcal{S} \Vdash \varphi$ iff $[\![\varphi]\!]^{\mathcal{S}} = W$ iff $W \subseteq [\![\varphi]\!]^{\mathcal{S}}$ iff $\top_{\mathbb{A}_{\mathcal{S}}} \subseteq [\![\varphi]\!]_{\mathbb{A}_{\mathcal{S}}}$ iff $[\![\top]\!]_{\mathbb{A}_{\mathcal{S}}} \subseteq [\![\varphi]\!]_{\mathbb{A}_{\mathcal{S}}}$ iff $\langle [\![\top]\!]_{\mathbb{A}_{\mathcal{S}}}, [\![\varphi]\!]_{\mathbb{A}_{\mathcal{S}}} \rangle \in \leq_{\mathbb{A}_{\mathcal{S}}}$ iff $\mathbb{A}_{\mathcal{S}} \Vdash \top \leq \varphi$. □

Proposition 3.2.9. Given a general structure $\mathcal{S}$ for non-normal modal logic, an unbound variable assignment β over $\mathbb{A}_{\mathcal{S}}$ such that $\beta(\mathbf{y}_1) = \{w\}$ and a closed simple formula φ, $\mathcal{S}, w \Vdash \varphi$ iff $\mathbb{A}_{\mathcal{S}}\beta \Vdash \mathbf{y}_1 \leq \varphi$.

Proof. $\mathcal{S}, w \Vdash \varphi$ iff $w \in [\![\varphi]\!]^{\mathcal{S}}$ iff $\{w\} \subseteq [\![\varphi]\!]^{\mathcal{S}}$ iff $\beta(\mathbf{y}_1) \subseteq [\![\varphi]\!]_{\mathbb{A}_{\mathcal{S}}}$ iff $[\![\mathbf{y}_1]\!]_{\mathbb{A}_{\mathcal{S}}} \subseteq [\![\varphi]\!]_{\mathbb{A}_{\mathcal{S}}}$ iff $\langle [\![\mathbf{y}_1]\!]_{\mathbb{A}_{\mathcal{S}}}, [\![\varphi]\!]_{\mathbb{A}_{\mathcal{S}}} \rangle \in \leq_{\mathbb{A}_{\mathcal{S}}}$ iff $\mathbb{A}_{\mathcal{S}}\beta \Vdash \mathbf{y}_1 \leq \varphi$. □

So, from a class of structures for non-normal modal logics, global and local entailment are preserved by the algebras of Definition 3.2.6.

Theorem 3.2.10. Let $\mathtt{S}$ be a class of general structures for non-normal modal logics, $\mathcal{A}(\mathtt{S}) = \{\mathbb{A}_{\mathcal{S}} : \mathcal{S} \in \mathtt{S}\}$ and $\psi_1, \ldots, \psi_k, \varphi$ closed simple formulas. Then

i. $\psi_1, \ldots, \psi_k \vDash^g_{\mathtt{S}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^g_{\mathcal{A}(\mathtt{S})} \varphi$,

ii. $\psi_1, \ldots, \psi_k \vDash^l_{\mathtt{S}} \varphi$ iff $\psi_1, \ldots, \psi_k \vDash^l_{\mathcal{A}(\mathtt{S})} \varphi$.

Proof. (i.) $\psi_1,\dots,\psi_k \vDash^g_{\mathtt{S}} \varphi$ iff, for every $\mathcal{S} \in \mathtt{S}$, $\mathcal{S} \Vdash \varphi$ whenever $\mathcal{S} \Vdash \psi_i$ for $i = 1,\dots,k$ iff (by Proposition 3.2.8), for every $\mathbb{A} \in \mathcal{A}(\mathtt{S})$, $\mathbb{A} \Vdash \top \leq \varphi$ whenever $\mathbb{A} \Vdash \top \leq \psi_i$ for $i = 1,\dots,k$ iff $\psi_1,\dots,\psi_k \Vdash^g_{\mathcal{A}(\mathtt{S})} \varphi$.

(ii.) $\psi_1,\dots,\psi_k \vDash^l_{\mathtt{S}} \varphi$ iff, for every $\mathcal{S} \in \mathtt{S}$ and $w \in W$, $\mathcal{S}, w \Vdash \varphi$ whenever $\mathcal{S}, w \Vdash \psi_i$ for $i = 1,\dots,k$ iff (by Proposition 3.2.9), for every $\mathbb{A} \in \mathcal{A}(\mathtt{S})$ and every assignment β such that $\beta(\mathbf{y}_1) = \{w\}$, $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \varphi$ whenever $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \psi_i$ for $i = 1,\dots,k$ iff, for every $\mathbb{A} \in \mathcal{A}(\mathtt{S})$ and every assignment β, $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \varphi$ whenever $\mathbb{A}\beta \Vdash \mathbf{y}_1 \leq \psi_i$ for $i = 1,\dots,k$ and $\mathbb{A}\beta \Vdash \Omega\mathbf{y}_1$ iff $\psi_1,\dots,\psi_k \Vdash^l_{\mathcal{A}(\mathtt{S})} \varphi$. □

Theorem 3.2.11. For every general structure $\mathcal{S}$ for non-normal modal logic, its corresponding Σ_N-algebra $\mathbb{A}_{\mathcal{S}}$ is appropriate for each rule in $\mathcal{R}_N$.

Proof. For structural and order rules the proof is the same of the case of induced algebras by general Kripke structures in normal modal logics given in [MSSV04]. In the following, let ρ be any ground substitution and α be an arbitrary unbound assignment over $\mathbb{A}_{\mathcal{S}}$. For the specific rules, it is analysed here first the case of eq**F**.

Assume that $\mathbb{A}_{\mathcal{S}}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \sqsubseteq \tau_2\rho$ and $\mathbb{A}_{\mathcal{S}}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho, \tau_2\rho \sqsubseteq \tau_1\rho$, that is, for every bound variable assignment β over $\mathbb{A}_{\mathcal{S}}$ there are:

- $\delta_1 \in \overline{\Gamma_1\rho} \cup \Gamma_2\rho \cup \{\tau_1\rho \sqsubseteq \tau_2\rho\}$ such that $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \delta_1$; and
- $\delta_2 \in \overline{\Gamma_1\rho} \cup \Gamma_2\rho \cup \{\tau_2\rho \sqsubseteq \tau_1\rho\}$ such that $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \delta_2$.

For each β there are two cases to be considered:

i. $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \delta_i$ with $\delta_i \in \overline{\Gamma_1\rho} \cup \Gamma_2\rho$ for some $i = 1, 2$ which immediately establishes the conclusion of the rule;

ii. otherwise, $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \tau_1\rho \sqsubseteq \tau_2\rho$ and $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \tau_2\rho \sqsubseteq \tau_1\rho$. Let $W', W'' \subseteq W$ be such that $[\![\tau_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = W'$ and $[\![\tau_2\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = W''$, thus $W' \subseteq W''$ and $W'' \subseteq W'$, that is, $W' = W''$. So $\mathbf{F}_{\mathbb{A}_{\mathcal{S}}}([\![\tau_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta}) = \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}(W') = \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}(W'') \subseteq \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}(W'') = \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}([\![\tau_2\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta})$ which establishes that $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \mathbf{F}(\tau_1\rho) \sqsubseteq \mathbf{F}(\tau_2\rho)$ and, so, the conclusion of the rule holds.

Now it is analysed the case of the rule R□. The proof for L□ is similar.

Assume that $\mathbb{A}_{\mathcal{S}}\alpha \Vdash \Gamma_1\rho \to \Gamma_2\rho, \tau_1\rho \sqsubseteq \mathbf{F}(\#\xi_1\rho)$. So, for every bound variable assignment β, there is $\delta_\beta \in \overline{\Gamma_1\rho} \cup \Gamma_2\rho \cup \{\tau_1\rho \sqsubseteq \mathbf{F}(\#\xi_1\rho)\}$ such that $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \delta_\beta$. For each β, two cases have to be considered:

- there is δ_β in $\overline{\Gamma_1\rho} \cup \Gamma_2\rho$ such that $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \delta_\beta$ which immediately establishes $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \Gamma_1\rho \to \Gamma_2\rho$ and, hence, the conclusion of the rule holds;

- otherwise δ_β is $\tau_1\rho \sqsubseteq \mathbf{F}(\#\xi_1\rho)$, that is, $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \tau_1\rho \sqsubseteq \mathbf{F}(\#\xi_1\rho)$. Suppose $[\![\tau_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = W'$ for a $W' \subseteq W$. Thus $W' \subseteq \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}(\#_{\mathbb{A}_{\mathcal{S}}}[\![\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha})$. Since $\#_{\mathbb{A}_{\mathcal{S}}} = \lambda a.a$, $W' \subseteq \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}([\![\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha})$, that is, $[\![\tau_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} \subseteq \Box_{\mathbb{A}_{\mathcal{S}}}([\![\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha})$ which establishes that $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \tau_1\rho \leq \Box\xi_1\rho$ and, so, the conclusion of the rule holds.

□

Given a Σ_N-algebra $\mathbb{A}$, there also is an induced general structure such that entailment is preserved as shown in the sequel. In what follows given $a \in T$, $\langle a\rangle_{\mathbb{A}} = \{a' \in \Omega_{\mathbb{A}} : a' \sqsubseteq_{\mathbb{A}} a\}$, and given $b \in F$, $\langle b\rangle_{\mathbb{A}} = \{a \in \Omega_{\mathbb{A}} : a \leq_{\mathbb{A}} b\}$.

Definition 3.2.12. Given a Σ_N-algebra $\mathbb{A} = \langle F, T, \cdot_{\mathbb{A}}\rangle$ appropriate for a sequent calculus for a non-normal modal logic, the *general structure induced by* $\mathbb{A}$ is $\mathcal{S}_{\mathbb{A}} = \langle W, f, A, V\rangle$ where:

- $W = \Omega_{\mathbb{A}}$;
- $f(X) = \begin{cases} \langle \mathbf{F}_{\mathbb{A}}(a)\rangle_{\mathbb{A}} & \text{if there is } a \in T \text{ such that } X = \langle a\rangle_{\mathbb{A}} \\ \varnothing & \text{otherwise} \end{cases}$, for $X \in 2^{\Omega_{\mathbb{A}}}$;
- $A = \{\langle b\rangle_{\mathbb{A}} : b \in F\}$;
- $V(\mathbf{p_i}) = \langle \mathbf{p_i}_{\mathbb{A}}\rangle_{\mathbb{A}}$. ■

In the sequel, $\langle t\rangle_{\mathbb{A}}$ will be denoted by $\langle t\rangle$ when the underlying algebra is clear from the context.

It remains to prove that $f(X)$ is well defined.

Lemma 3.2.13. If $a, a' \in T$ are such that $\langle a\rangle = \langle a'\rangle$ then $\langle \mathbf{F}_{\mathbb{A}}(a)\rangle = \langle \mathbf{F}_{\mathbb{A}}(a')\rangle$.

Proof. Since $\mathbb{A}$ is appropriate and

$$\frac{\Omega\tau_0, \tau_0 \sqsubseteq \tau_1, \Gamma_1 \to \Gamma_2, \tau_0 \sqsubseteq \tau_2 \quad \Omega\tau_0, \tau_0 \sqsubseteq \tau_2, \Gamma_1 \to \Gamma_2, \tau_0 \sqsubseteq \tau_1}{\Gamma_1 \to \Gamma_2, \mathbf{F}(\tau_1) \sqsubseteq \mathbf{F}(\tau_2)} \lhd \tau_0 : \mathbf{y}, \tau_0 \notin \tau_1, \tau_2, \Gamma_1, \Gamma_2$$

is a derived rule (using rules eq$\mathbf{F}$ and RgenT), then, by soundness and using the appropriate assignments, $\mathbf{F}_{\mathbb{A}}(a) \sqsubseteq_{\mathbb{A}} \mathbf{F}_{\mathbb{A}}(a')$ and $\mathbf{F}_{\mathbb{A}}(a') \sqsubseteq_{\mathbb{A}} \mathbf{F}_{\mathbb{A}}(a)$ and so, since the sequents

$$\Omega\mathbf{y}_1, \mathbf{F}(a) \sqsubseteq \mathbf{F}(a'), \mathbf{y}_1 \sqsubseteq \mathbf{F}(a) \to \mathbf{y}_1 \sqsubseteq \mathbf{F}(a')$$

and

$$\Omega\mathbf{y}_1, \mathbf{F}(a') \sqsubseteq \mathbf{F}(a), \mathbf{y}_1 \sqsubseteq \mathbf{F}(a') \to \mathbf{y}_1 \sqsubseteq \mathbf{F}(a)$$

are derivable (both by transT rule), then $\langle \mathbf{F}_{\mathbb{A}}(a)\rangle = \langle \mathbf{F}_{\mathbb{A}}(a')\rangle$. □

To show the preservation of entailment first it is showed that the structure previously defined is, in fact, a general non-normal structure.

Proposition 3.2.14. Given a Σ_N-algebra $\mathbb{A}$ appropriate for a sequent calculus $\mathcal{C}_N$ for a non-normal modal logic, the tuple $\mathcal{S}_{\mathbb{A}}$ of Definition 3.2.12 is a general structure for a non-normal modal logic.

Proof. It has to be shown that W is non-empty and A satisfies the conditions given in 3.2.5.

(A) W is non-empty. In fact, by absurd, assume that $W = \Omega_{\mathbb{A}} = \varnothing$. Since $\mathbb{A}$ is appropriate for $\mathcal{C}_N$, it is so for rules cons and Ω, then $\top_{\mathbb{A}} \in \Omega_{\mathbb{A}}$. Therefore $W \neq \varnothing$.

(B) $\varnothing \in A$. Indeed, $\varnothing = \langle [\![\mathbf{f}]\!]_{\mathbb{A}}\rangle$ since $\mathbb{A}$ is appropriate for rule R$\mathbf{f}$.

(C) A is closed for unions. It has to be shown that if $\langle b\rangle, \langle b'\rangle \in A$ then $(\langle b\rangle \cup \langle b'\rangle) \in A$. Take the set

$$\langle \vee_{\mathbb{A}}(b, b')\rangle = \{a \in W : a \leq_{\mathbb{A}} \vee_{\mathbb{A}}(b, b')\}$$

which is in A. The following facts show that

$$\langle \vee_{\mathbb{A}}(b, b')\rangle = \{a \in W : a \leq_{\mathbb{A}} b \text{ or } a \leq_{\mathbb{A}} b'\}$$

where the latter set is precisely $\langle b\rangle \cup \langle b'\rangle$.

(i) $\{a \in W : a \leq_{\mathbb{A}} b \text{ or } a \leq_{\mathbb{A}} b'\} \subseteq \{a \in W : a \leq_{\mathbb{A}} \vee_{\mathbb{A}}(b, b')\}$. Assume that $a \leq_{\mathbb{A}} b$ or $a \leq_{\mathbb{A}} b'$ for $a \in \Omega_{\mathbb{A}}$. By satisfaction, choosing α such that $\alpha(\mathbf{x_1}) = a$, $\alpha(\mathbf{z_1}) = b$ and $\alpha(\mathbf{z_2}) = b'$ it holds $\mathbb{A}\alpha \Vdash \Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2}$. Observe that

$$\Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2} \vdash_{\mathcal{C}_N} \Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1} \vee \mathbf{z_2}$$

by an application of rule R$\vee$. Since the calculus is sound (by Theorem 2.5.5),

$$\Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2} \vDash_{\mathcal{A}(\mathcal{C}_N)} \Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1} \vee \mathbf{z_2}$$

where $\mathcal{A}(\mathcal{C}_N)$ is the class of the algebras appropriate for $\mathcal{C}_N$. Hence, $\mathbb{A}\alpha \Vdash \Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1} \vee \mathbf{z_2}$. Again by satisfaction and taking into account the choice of α, $a \leq_{\mathbb{A}} \vee_{\mathbb{A}}(b, b')$.

(ii) $\{a \in W : a \leq_{\mathbb{A}} \vee_{\mathbb{A}}(b, b')\} \subseteq \{a \in W : a \leq_{\mathbb{A}} b \text{ or } a \leq_{\mathbb{A}} b'\}$. The proof is similar taking into account that $\Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1} \vee \mathbf{z_2} \vdash_{\mathcal{C}_N} \Omega\mathbf{x_1} \to \mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2}$ as showed below:

1.	$\Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2}$	cutF 2,3
2.	$\Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2}, \mathbf{x_1} \leq \mathbf{z_1} \vee \mathbf{z_2}$	RwF 4
3.	$\mathbf{x_1} \leq \mathbf{z_1} \vee \mathbf{z_2}, \Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2}$	L$\vee$ 5,6
4.	$\Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \mathbf{z_1} \vee \mathbf{z_2}$	hyp
5.	$\mathbf{x_1} \leq \mathbf{z_1}, \Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2}$	axiom
6.	$\mathbf{x_1} \leq \mathbf{z_2}, \Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_2}$	axiom

(D) A is closed for complements. It has to be shown that if $\langle b\rangle \in A$ then $(W \setminus \langle b\rangle) \in A$. Take the set

$$\langle \underset{\mathbb{A}}{\neg}(b)\rangle = \{a \in W : a \leq_{\mathbb{A}} \underset{\mathbb{A}}{\neg}(b)\}$$

which is in A. The following facts show that

$$\langle \underset{\mathbb{A}}{\neg}(b)\rangle = \{a \in W : a \not\leq_{\mathbb{A}} b\}$$

where the latter set is precisely $W \setminus \langle b\rangle$.

(i) $\{a \in W : a \not\leq_{\mathbb{A}} b\} \subseteq \{a \in W : a \leq_{\mathbb{A}} \neg_{\mathbb{A}}(b)\}$. The proof is similar to the case (i) of (C) taking into account that $\Omega\mathbf{x_1} \rightarrow \mathbf{x_1} \not\leq \mathbf{z_1} \vdash_{\mathcal{C}_N} \Omega\mathbf{x_1} \rightarrow \mathbf{x_1} \leq \neg\,\mathbf{z_1}$ by application of rules R$\neg$ and LxeF.

(ii) $\{a \in W : a \leq_{\mathbb{A}} \neg_{\mathbb{A}}(b)\} \subseteq \{a \in W : a \not\leq_{\mathbb{A}} b\}$. Again the proof is similar taking into account $\Omega\mathbf{x_1} \rightarrow \mathbf{x_1} \leq \neg\,\mathbf{z_1} \vdash_{\mathcal{C}_N} \Omega\mathbf{x_1} \rightarrow \mathbf{x_1} \not\leq \mathbf{z_1}$ as showed below:

1.	$\Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \not\leq \mathbf{z_1}$	cutF 2,3
2.	$\Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \not\leq \mathbf{z_1}, \mathbf{x_1} \leq \neg\,\mathbf{z_1}$	RwF 4
3.	$\mathbf{x_1} \leq \neg\,\mathbf{z_1}, \Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \not\leq \mathbf{z_1}$	L$\neg$ 5
4.	$\Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \neg\,\mathbf{z_1}$	hyp
5.	$\Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \not\leq \mathbf{z_1}, \mathbf{x_1} \leq \mathbf{z_1}$	RxiF 6
6.	$\mathbf{x_1} \leq \mathbf{z_1}, \Omega\mathbf{x_1}$	$\rightarrow$	$\mathbf{x_1} \leq \mathbf{z_1}$	axiom

(E) A is closed under f. It has to be shown that if $\langle b\rangle \in A$ then $f(\langle b\rangle) \in A$. From $\langle b\rangle \in \mathbb{A}$ it can be seen that there is $X \in 2^{\Omega_{\mathbb{A}}}$ such that $X = \langle b\rangle$, so

$$f(\langle b\rangle) = \langle \mathbf{F}_{\mathbb{A}}(b)\rangle = \{a \in W : a \sqsubseteq_{\mathbb{A}} \mathbf{F}_{\mathbb{A}}(b)\}$$

. The following facts show that the latter set is equal to

$$\{a \in W : a \leq_{\mathbb{A}} \Box_{\mathbb{A}}(b)\} = \langle \Box_{\mathbb{A}}(b)\rangle$$

which is in A.

(i) $\{a \in W : a \sqsubseteq_{\mathbb{A}} \mathbf{F}_{\mathbb{A}}(b)\} \subseteq \{a \in W : a \leq_{\mathbb{A}} \Box_{\mathbb{A}}(b)\}$. The proof is similar to the case (i) of (C) taking into account that $\Omega\mathbf{x_1} \rightarrow \mathbf{x_1} \sqsubseteq \mathbf{F}(\#\mathbf{z_1}) \vdash_{\mathcal{C}_N} \Omega\mathbf{x_1} \rightarrow \mathbf{x_1} \leq \Box\mathbf{z_1}$ by applying the rule R$\Box$.

(ii) $\{a \in W : a \leq_{\mathbb{A}} \Box_{\mathbb{A}}(b)\} \subseteq \{a \in W : a \sqsubseteq_{\mathbb{A}} \mathbf{F}_{\mathbb{A}}(b)\}$. Again the proof is similar taking into account that $\Omega\mathbf{x_1} \to \mathbf{x_1} \leq \Box\mathbf{z_1} \vdash_{\mathcal{C}_N} \Omega\mathbf{x_1} \to \mathbf{x_1} \sqsubseteq \mathbf{F}(\#\mathbf{z_1})$ by the following derivation:

1.	$\Omega\mathbf{x_1}$	$\to$	$\mathbf{x_1} \sqsubseteq \mathbf{F}(\#\mathbf{z_1})$	cutF 2,3
2.	$\Omega\mathbf{x_1}$	$\to$	$\mathbf{x_1} \sqsubseteq \mathbf{F}(\#\mathbf{z_1}), \mathbf{x_1} \leq \Box\mathbf{z_1}$	RwF 4
3.	$\mathbf{x_1} \leq \Box\mathbf{z_1}, \Omega\mathbf{x_1}$	$\to$	$\mathbf{x_1} \sqsubseteq \mathbf{F}(\#\mathbf{z_1})$	L$\Box$ 5
4.	$\Omega\mathbf{x_1}$	$\to$	$\mathbf{x_1} \leq \Box\mathbf{z_1}$	hyp
5.	$\mathbf{x_1} \sqsubseteq \mathbf{F}(\#\mathbf{z_1}), \Omega\mathbf{x_1}$	$\to$	$\mathbf{x_1} \sqsubseteq \mathbf{F}(\#\mathbf{z_1})$	axiom

□

The following results establish the preservation of the semantics of closed simple formulas when moving to general non-normal structures from the bi-sorted algebras appropriate for a sequent calculus for a non-normal modal logic. It is given now a lemma about the denotation of this kind of formulas.

Lemma 3.2.15. Let $\mathbb{A}$ be a Σ_N-algebra appropriate for $\mathcal{C}_N$. Then, for every closed simple formula φ, $\langle[\![\varphi]\!]_{\mathbb{A}}\rangle = [\![\varphi]\!]^{\mathcal{S}_{\mathbb{A}}}$.

Proof. The proof is by structural induction on φ.

- φ is $\mathbf{f}$. Then $\langle[\![\mathbf{f}]\!]_{\mathbb{A}}\rangle = \langle\mathbf{f}_{\mathbb{A}}\rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \mathbf{f}_{\mathbb{A}}\}$. Since $\mathbb{A}$ is appropriate for rule R$\mathbf{f}$ the latter set coincides with $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \perp_{\mathbb{A}}\}$. Moreover, this set is equal to $\varnothing$ since $\mathbb{A}$ is also appropriate for rule $\Omega\perp$. Therefore $\langle[\![\mathbf{f}]\!]_{\mathbb{A}}\rangle = [\![\mathbf{f}]\!]^{\mathcal{S}_{\mathbb{A}}}$.
- φ is $\mathbf{p}_i$. Then $\langle[\![\mathbf{p}_i]\!]_{\mathbb{A}}\rangle = \langle\mathbf{p}_{i\mathbb{A}}\rangle = V(\mathbf{p}_i) = [\![\mathbf{p}_i]\!]^{\mathcal{S}_{\mathbb{A}}}$.
- φ is $\neg\varphi'$. Then $\langle[\![\neg\varphi']\!]_{\mathbb{A}}\rangle = \langle\neg_{\mathbb{A}}[\![\varphi']\!]_{\mathbb{A}}\rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \neg_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}})\}$ which coincides with $\{u \in \Omega_{\mathbb{A}} : u \not\leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}}\}$ as seen in part (D) of the proof of Proposition 3.2.14. Therefore,

  $$\langle[\![\neg\varphi']\!]_{\mathbb{A}}\rangle = \Omega_{\mathbb{A}} \setminus \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}}\} = \Omega_{\mathbb{A}} \setminus \langle[\![\varphi']\!]_{\mathbb{A}}\rangle.$$

  By the induction hypothesis, the latter is equal to $\Omega_{\mathbb{A}} \setminus [\![\varphi']\!]^{\mathcal{S}_{\mathbb{A}}} = [\![\neg\varphi']\!]^{\mathcal{S}_{\mathbb{A}}}$.
- φ is $\Box\varphi'$. Then $\langle[\![\Box\varphi']\!]_{\mathbb{A}}\rangle = \langle\Box_{\mathbb{A}}[\![\varphi']\!]_{\mathbb{A}}\rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} \Box_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}})\}$ which coincides with $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} f([\![\varphi']\!]_{\mathbb{A}})\}$ as seen in part (E) of the proof of Proposition 3.2.14. Therefore,

  $$\langle[\![\Box\varphi']\!]_{\mathbb{A}}\rangle = f(\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}}\}).$$

  By the induction hypothesis, the latter is equal to $f([\![\varphi']\!]^{\mathcal{S}_{\mathbb{A}}})$ which is $[\![\Box\varphi']\!]^{\mathcal{S}_{\mathbb{A}}}$.

- φ is $\varphi' \vee \varphi''$. Then $\langle [\![\varphi' \vee \varphi'']\!]_{\mathbb{A}} \rangle = \langle \vee_{\mathbb{A}}([\![\varphi']\!]_{\mathbb{A}}, [\![\varphi'']\!]_{\mathbb{A}}) \rangle$ which coincides with $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}} \text{ or } u \leq_{\mathbb{A}} [\![\varphi'']\!]_{\mathbb{A}}\}$ as seen in part (C) of Proposition 3.2.14. Therefore,

$$\langle [\![\varphi' \vee \varphi'']\!]_{\mathbb{A}} \rangle = \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi']\!]_{\mathbb{A}}\} \cup \{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi'']\!]_{\mathbb{A}}\}.$$

  By the induction hypothesis, the latter is equal to $[\![\varphi']\!]^{\mathcal{S}_{\mathbb{A}}} \cup [\![\varphi'']\!]^{\mathcal{S}_{\mathbb{A}}} = [\![\varphi' \vee \varphi'']\!]^{\mathcal{S}_{\mathbb{A}}}$.

□

Now it is shown the preservation of satisfaction for closed simple formulas in order to have results for local and global entailment.

Proposition 3.2.16. Given a Σ_N-algebra $\mathbb{A}$ appropriate for a sequent calculus $\mathcal{C}_N$ for a non-normal modal logic and a closed simple formula φ, $\mathbb{A} \Vdash \top \leq \varphi$ iff $\mathcal{S}_{\mathbb{A}} \Vdash \varphi$.

Proof. Assume $\mathbb{A} \Vdash \top \leq \varphi$. Then $\top_{\mathbb{A}} \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$. Since $\mathbb{A}$ is appropriate for rules $\top$ and transF, for every $t \in T$, $t \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$. In particular for every $u \in \Omega_{\mathbb{A}}$, $u \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$. Therefore, $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}\} = \Omega_{\mathbb{A}}$. So $\langle [\![\varphi]\!]_{\mathbb{A}} \rangle = \Omega_{\mathbb{A}}$ and $[\![\varphi]\!]^{\mathcal{S}_{\mathbb{A}}} = W$ by Lemma 3.2.15, that is, $\mathcal{S}_{\mathbb{A}} \Vdash \varphi$.

For the other direction, assume $\mathcal{S}_{\mathbb{A}} \Vdash \varphi$, then $\{u \in \Omega_{\mathbb{A}} : u \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}\} = \Omega_{\mathbb{A}}$. Since $\mathbb{A}$ is appropriate for rule $\Omega\top$, $\top_{\mathbb{A}} \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$ so $\mathbb{A} \Vdash \top \leq \varphi$. □

Proposition 3.2.17. Given a Σ_N-algebra $\mathbb{A}$ appropriate for $\mathcal{C}_N$, assignments α and β over $\mathbb{A}$ such that $\beta(\mathbf{y_1}) = u \in \Omega_{\mathbb{A}}$ and a closed simple formula φ, $\mathbb{A}\alpha\beta \Vdash \mathbf{y_1} \leq \varphi$ iff $\mathcal{S}_{\mathbb{A}}, u \Vdash \varphi$.

Proof. Observe that $\mathbb{A}\alpha\beta \Vdash \mathbf{y_1} \leq \varphi$ iff $\beta(\mathbf{y_1}) \leq_{\mathbb{A}} [\![\varphi]\!]_{\mathbb{A}}$ iff $u \in \langle [\![\varphi]\!]_{\mathbb{A}} \rangle$ iff $u \in [\![\varphi]\!]^{\mathcal{S}_{\mathbb{A}}}$ iff $\mathcal{S}_{\mathbb{A}}, u \Vdash \varphi$. □

Finally, the theorem about the equivalence between entailments.

Theorem 3.2.18. Given a class $\mathcal{A}$ of Σ_N-algebras appropriate for a sequent calculus $\mathcal{C}_N$ for a non-normal modal logic and $\psi_1, \ldots, \psi_k, \varphi$ closed simple formulas, let $\mathtt{S}_{\mathcal{A}} = \{\mathcal{S}_{\mathbb{A}} : \mathbb{A} \in \mathcal{A}\}$, then for $d = l, g$:

$$\psi_1, \ldots, \psi_k \vDash^d_{\mathcal{A}} \varphi \text{ iff } \psi_1, \ldots, \psi_k \vDash^d_{\mathtt{S}_{\mathcal{A}}} \varphi.$$

Proof. The proof is immediate using Propositions 3.2.16 and 3.2.17. □

The smallest non-normal modal system is denoted as E in [Che80] and the elements of class of all systems in which the semantics is given by minimal models is called *classical systems* [Seg71]. Two other classes of non-normal modal systems are referred in the literature: the class of *monotonic systems*, consisted of those in which the rule

$$\frac{\varphi_1 \supset \varphi_2}{\Box\varphi_1 \supset \Box\varphi_2}$$

is admissible (that is, every formula that can be derived using this rule is already derivable in the system without it), and the class of *regular systems*, consisted of systems in which the rule

$$\frac{(\varphi_1 \wedge \varphi_2) \supset \varphi_3}{(\Box\varphi_1 \wedge \Box\varphi_2) \supset \Box\varphi_3}$$

is admissible. It is known that the following inclusions hold:

Normal systems $\subseteq$ Regular systems $\subseteq$ Monotonic systems $\subseteq$ Classical systems.

In the following, labelled non-normal modal logic systems are defined. The classical systems are characterized by the smallest set of rules of the next definition.

Definition 3.2.19. A *labelled non-normal modal logic system* is a relational logic system $\mathcal{L}_N = \langle \Sigma_N, \mathcal{R}_N, \mathcal{A}\rangle$ where the signature Σ_N is as Definition 3.2.2, the rules are at least those described in Definition 3.2.3 and $\mathcal{A}$ is a class of Σ_N-algebras as in Definition 3.2.6. ■

The examples in the sequel characterize the other classes mentioned above as logic systems labelled with truth values. First, the monotonic systems.

Example 3.2.20. Consider the non-normal modal logic system $\mathcal{MT} = \langle \Sigma_{MT}, \mathcal{R}_{MT}, \mathcal{A}_{MT}\rangle$ corresponding to monotonic systems, see [Che80], where:

- Σ_{MT} is a non-normal modal signature as in Definition 3.2.2;
- $\mathcal{R}_{MT}$ is the set of rules as in Definition 3.2.3 plus the rule M:

$$\frac{}{\Omega\tau_1, \#\xi_1 \sqsubseteq \#\xi_2, \tau_1 \sqsubseteq \mathbf{F}(\#\xi_1), \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{F}(\#\xi_2)}$$

- $\mathcal{A}_{MT}$ is a class of Σ_{MT}-algebras as in Definition 3.2.6 full for $\langle \Sigma_{MT}, \mathcal{R}_{MT}\rangle$ corresponding to supplemented structures, i.e., to structures where the condition *if* $f(X \cap Y)$ *then* $f(X)$ *and* $f(Y)$ holds. ■

In a system equipped with the rule M presented in the Example above, the rule

$$\frac{\varphi_1 \supset \varphi_2}{\Box\varphi_1 \supset \Box\varphi_2}$$

is globally derived as it can be seen by the following derivation.

1.		$\rightarrow$	$\top \leq \Box\varphi_1 \supset \Box\varphi_2$	RgenF 2
2.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top$	$\rightarrow$	$\mathbf{y_1} \leq \Box\varphi_1 \supset \Box\varphi_2$	R$\supset$ 3
3.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \leq \Box\varphi_1$	$\rightarrow$	$\mathbf{y_1} \leq \Box\varphi_2$	R,L$\Box$ 4
4.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	cutT 5,6
5.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2$	cutF 7,8
6.	$\#\varphi_1 \sqsubseteq \#\varphi_2, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	M
7.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2,$ $\#\varphi_1 \leq \varphi_1 \supset \varphi_2$	transF 9,10
8.	$\#\varphi_1 \leq \varphi_1 \supset \varphi_2, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2$	L$\supset$ 11,12
9.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2,$ $\#\varphi_1 \sqsubseteq \top$	$\top$
10.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2,$ $\top \leq \varphi_1 \supset \varphi_2$	hyp
11.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2,$ $\#\varphi_1 \leq \varphi_1$	cutT 13,14
12.	$\#\varphi_1 \leq \varphi_2, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2$	R# 16
13.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2,$ $\#\varphi_1 \leq \varphi_1, \#\varphi_1 \sqsubseteq \#\varphi_1$	ref
14.	$\#\varphi_1 \sqsubseteq \#\varphi_1, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2,$ $\#\varphi_1 \leq \varphi_1$	L# 15
15.	$\#\varphi_1 \leq \varphi_1, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \sqsubseteq \#\varphi_2,$ $\#\varphi_1 \leq \varphi_1$	axiom
16.	$\#\varphi_1 \leq \varphi_2, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2), \#\varphi_1 \leq \varphi_2$	axiom

Example 3.2.21. Consider the non-normal modal logic system $\mathcal{R}eg = \langle \Sigma_R, \mathcal{R}_R, \mathcal{A}_R \rangle$ corresponding to regular systems, see [Che80], where:

- Σ_R is a non-normal modal signature as in Definition 3.2.2;
- $\mathcal{R}_R$ is the set of rules described in Definition 3.2.3 plus the rules M (presented in Example 3.2.20) and C:

$$\frac{}{\Omega\tau_1, \tau_1 \sqsubseteq \mathbf{F}(\#\xi_1), \tau_1 \sqsubseteq \mathbf{F}(\#\xi_2), \Gamma_1 \rightarrow \Gamma_2, \tau_1 \sqsubseteq \mathbf{F}(\#(\xi_1 \wedge \xi_2))}$$

- $\mathcal{A}_R$ is a class of Σ_R-algebras as in Definition 3.2.6 full for $\langle \Sigma_R, \mathcal{R}_R \rangle$ corresponding to the class of quasi-filters. A structure belongs to the class of quasi-filters if it is supplemented and if it is closed under intersections. ■

In system equipped with rules M and C, presented in the Example 3.2.21, the rule

$$\frac{(\varphi_1 \wedge \varphi_2) \supset \varphi_3}{(\Box\varphi_1 \wedge \Box\varphi_2) \supset \Box\varphi_3}$$

is globally derived as it can be seen by the following derivation.

1.		$\to$	$\top \leq (\Box\varphi_1 \wedge \Box\varphi_2) \supset \Box\varphi_3$	RgenF 2
2.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top$	$\to$	$\mathbf{y_1} \leq (\Box\varphi_1 \wedge \Box\varphi_2) \supset \Box\varphi_3$	R$\supset$ 3
3.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \leq \Box\varphi_1 \wedge \Box\varphi_2$	$\to$	$\mathbf{y_1} \leq \Box\varphi_3$	L$\wedge$ + R$\Box$ 4
4.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \leq \Box\varphi_1, \mathbf{y_1} \leq \Box\varphi_2$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3)$	L$\Box$ + L$\Box$ 5
5.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3)$	cutF 6,7
6.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \leq (\varphi_1 \wedge \varphi_2) \supset \varphi_3$	transF 8,9
7.	$\#(\varphi_1 \wedge \varphi_2) \leq (\varphi_1 \wedge \varphi_2) \supset \varphi_3, \Omega\mathbf{y_1},$ $\mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3)$	L$\supset$ 10,11
8.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \sqsubseteq \top$	$\top$
9.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\top \leq (\varphi_1 \wedge \varphi_2) \supset \varphi_3$	weak's + hyp
10.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \leq (\varphi_1 \wedge \varphi_2)$	cutT 12,13
11.	$\#(\varphi_1 \wedge \varphi_2) \leq \varphi_3, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3)$	cutT 15,16
12.	$\Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \leq (\varphi_1 \wedge \varphi_2),$ $\#(\varphi_1 \wedge \varphi_2) \sqsubseteq \#(\varphi_1 \wedge \varphi_2)$	ref
13.	$\#(\varphi_1 \wedge \varphi_2) \sqsubseteq \#(\varphi_1 \wedge \varphi_2), \Omega\mathbf{y_1},$ $\mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \leq (\varphi_1 \wedge \varphi_2)$	L# 14
14.	$\#(\varphi_1 \wedge \varphi_2) \leq (\varphi_1 \wedge \varphi_2), \Omega\mathbf{y_1},$ $\mathbf{y_1} \sqsubseteq \top, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \leq (\varphi_1 \wedge \varphi_2)$	axiom
15.	$\#(\varphi_1 \wedge \varphi_2) \leq \varphi_3, \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#(\varphi_1 \wedge \varphi_2))$	C
16.	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#(\varphi_1 \wedge \varphi_2)), \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\#(\varphi_1 \wedge \varphi_2) \leq \varphi_3, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\to$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3)$	cutT 17,18

17.	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#(\varphi_1 \wedge \varphi_2)), \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\#(\varphi_1 \wedge \varphi_2) \leq \varphi_3, \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \sqsubseteq \#\varphi_3$	R# 19
18.	$\#(\varphi_1 \wedge \varphi_2) \sqsubseteq \#\varphi_3, \Omega\mathbf{y_1},$ $\#(\varphi_1 \wedge \varphi_2) \leq \varphi_3, \mathbf{y_1} \sqsubseteq \top,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#(\varphi_1 \wedge \varphi_2)),$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3)$	M
19.	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#(\varphi_1 \wedge \varphi_2)), \Omega\mathbf{y_1}, \mathbf{y_1} \sqsubseteq \top,$ $\#(\varphi_1 \wedge \varphi_2) \leq \varphi_3,$ $\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_1), \mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_2)$	$\rightarrow$	$\mathbf{y_1} \sqsubseteq \mathbf{F}(\#\varphi_3),$ $\#(\varphi_1 \wedge \varphi_2) \leq \varphi_3$	axiom

Finally, in the LSLTV approach an alternative definition of System K is presented by enriching the systems described in Example 3.2.21 with a rule inferring that the application of f to $\top$ contains $\top$ itself, that is, making $\Box\top$ a theorem of the system.

Example 3.2.22. Consider the system $\mathcal{RN} = \langle \Sigma_{RN}, \mathcal{R}_{RN}, \mathcal{A}_{RN} \rangle$ where:

- Σ_{RN} is a non-normal modal signature as in Definition 3.2.2;
- $\mathcal{R}_{RN}$ is the set of rules as in Example 3.2.21 plus the rule N:

$$\frac{}{\Gamma_1 \rightarrow \Gamma_2, \top \sqsubseteq \mathbf{F}(\top)}$$

- $\mathcal{A}_{RN}$ is a class of Σ_{RN}-algebras as in Definition 3.2.6 corresponded to the class of filters, full for $\langle \Sigma_{MT}, \mathcal{R}_{MT} \rangle$. A structure belongs to the class of filters if is a quasi-filter with unit.

The system $\mathcal{RN}$ is actually equivalent to the smallest normal modal logic system K. ■

Chellas [Che80] also mentions about the well-known axioms for the construction of more systems like $\mathbf{T}, \mathbf{4}, \mathbf{D}$ and gives the characterization of them in terms of minimal models. Table 3.6 presents the corresponding rules.

$$\frac{}{\Gamma_1 \rightarrow \Gamma_2, \mathbf{F}(\tau_1) \sqsubseteq \tau_1} \quad \mathrm{T_{NN}}$$

$$\frac{}{\Gamma_1 \rightarrow \Gamma_2, \mathbf{F}(\tau_1) \sqsubseteq \mathbf{F}(\mathbf{F}(\tau_1))} \quad \mathrm{4_{NN}}$$

$$\frac{}{\Omega\tau_1, \tau_1 \sqsubseteq \mathbf{F}(\#\xi_1), \Gamma_1 \rightarrow \Gamma_2, \tau_1 \not\sqsubseteq \mathbf{F}(\#\neg\xi_1)} \quad \mathrm{D_{NN}}$$

Table 3.6: Rules for **T**, **4** and **D** axioms in non-normal systems

As mentioned after Example 3.1.9, deontic logics could be defined as non-normal systems in order to avoid some theorems. In fact, deontic systems would be defined alternatively as a non-normal modal logic system of Definition 3.2.19 equipped with rules C, N and D_{NN}. Notice that this system is not normal and not even monotonic since the formula $\Box(\varphi_1 \wedge \varphi_2) \supset \Box\varphi_1 \wedge \Box\varphi_2$ is not derived from the rules.

Chapter 4

Fusion of Systems Labelled with Truth Values

Introduced by Thomason [Tho84], fusion is a well known way to combine normal modal logics. Basically it is defined simply as the union of the given modal axiomatic systems sharing propositional connectives, so the resulting system has no axiom that uses together the connectives that are not shared. Several results about transference of properties by fusion are known and can be found in [KW91, FS96, KW97, Wol98, Kur06].

The main aim of this chapter is to extend the concept of fusion to other kinds of systems besides normal modal logics. The chapter starts by describing an algebraic approach to the definition of fusion of Logic Systems Labelled with Truth Values following the original concept of the combination. Then the definition of fusion is generalized by defining it over the category of Logic Systems Labelled with Truth Values. To this purpose Section 4.2 starts by presenting logic system morphisms and some of their properties. Next, in Subsection 4.2.2 the categorical presentation of fusion is introduced as a colimit of two inclusions morphisms. This categorical definition in fact, besides having the algebraic fusion as a particular case, encompasses a much wider class of combinations besides fusions. Even for fusions, it allows the fusion of systems not previously considered. Finally in Section 4.3 preservation results are established, namely preservation of soundness by fusion as well as preservation of completeness under mild assumptions.

4.1 Algebraic Presentation of Fusion

In this section an algebraic account of fusion of (normal) modal logic systems based in [CCG$^+$08] is described. In this algebraic account, propositional connectives and operators and their rules are shared between the modal logic systems but modalities are not, following the original concept of fusion. In Subsection 4.2.2, the fusion of logic systems possibly sharing modalities is analysed.

Definition 4.1.1. Let $\mathcal{L}_1 = \langle \Sigma_1, \mathcal{R}_1, \mathcal{A}_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \mathcal{R}_2, \mathcal{A}_2 \rangle$ be two (normal) modal logic systems. The *fusion* of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the logic system $\langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ where:

- Σ is the signature $\langle C, O, \mathbb{P}, X, Y, Z \rangle$ where $C_0 = \{\mathbf{t}, \mathbf{f}\} \cup \{\mathbf{p}_i : i \in \mathbb{N}\}$; $C_1 = \{\neg\} \cup \{\Box'\} \cup \{\Box''\}$; $C_2 = \{\wedge, \vee, \supset\}$; $C_k = \varnothing$ for all $k \geq 3$; $O_0 = \{\top, \bot\}$; $O_1 = \{\mathbf{I}\} \cup \{\mathbf{N}'\} \cup \{\mathbf{N}''\}$; $O_2 = \{\mathbf{lb}\}$; $O_k = \varnothing$ for all $k \geq 3$; $\mathbb{P}_1 = \{\Omega\}$; $\mathbb{P}_2 = \{\sqsubseteq\}$; $\mathbb{P}_k = \varnothing$ for all $k \geq 3$;

- $\mathcal{R}$ keeps all the structural rules, the order rules and the logical rules for $\supset$, $\neg$, $\vee$, $\wedge$, **lb** and **I**. Furthermore, each rule in $\mathcal{L}_1$ ($\mathcal{L}_2$) that involves the connective $\Box$ or the operator **N** is in $\mathcal{R}$ by renaming the connective $\Box$ to $\Box'$ ($\Box''$) and the operator **N** to $\mathbf{N}'$ ($\mathbf{N}''$);

- $\mathcal{A}$ is the class of Σ-algebras $\{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2} : \mathbb{A}_1 \in \mathcal{A}_1, \mathbb{A}_2 \in \mathcal{A}_2, \mathbb{A}_1|_{\Sigma_{P+}} = \mathbb{A}_2|_{\Sigma_{P+}}\}$ where $\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}$ is such that $\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}|_{\Sigma_{P+}} = \mathbb{A}_1|_{\Sigma_{P+}}$, $\Box'_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \Box_{\mathbb{A}_1}, \Box''_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \Box_{\mathbb{A}_2}, \mathbf{N}'_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \mathbf{N}_{\mathbb{A}_1}$ and $\mathbf{N}''_{\mathbb{A}_{\mathbb{A}_1,\mathbb{A}_2}} = \mathbf{N}_{\mathbb{A}_2}$, where Σ_{P+} is the signature with the shared propositional connectives and operators. ■

Observe that $\mathcal{L}_1$ and $\mathcal{L}_2$ share all the connectives and operators except the modalities and the operator **N**. Moreover the rules of these systems are the same except the rules for the modal constructors and their **N** operators.

4.2 Categorical Presentation of Fusion

In this section a categorical description of fusion is given by showing that it is a colimit in $\mathcal{L}og$, the category of LSLTV presented in Subsection 4.2.1. In a general way, fusion is obtained as a pushout as illustrated by the diagram in Figure 4.1 where the morphisms are inclusions.

The section starts with LSLTV morphisms and some of their properties. Subsection 4.2.2 presents the definition of the envisaged combination of systems.

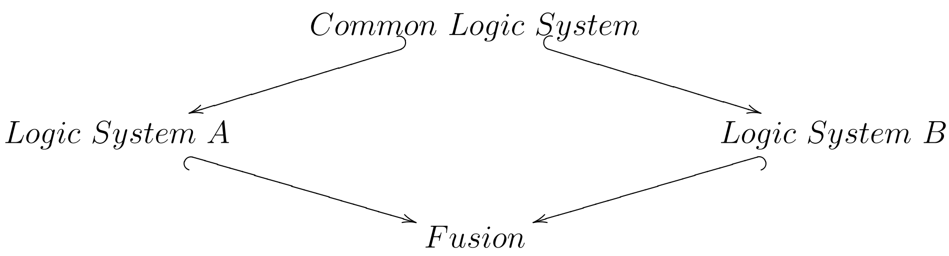

Figure 4.1: General diagram of fusion

4.2.1 Category of logic systems

In order to define fusion as a categorical operation, logic system morphisms are first introduced. They are a pair of maps, one map between signatures and the other (a contravariant map) between classes of algebras. In this work we assume that the set of relations ⅊ between terms is the same in all logical systems. The notion of morphism between signatures is the starting point of this subsection.

Definition 4.2.1. Given signatures Σ and Σ' with $\text{⅊} = \text{⅊}'$, a *signature morphism from* Σ *to* Σ' is a tuple $h = \langle h_C, h_O, h_X, h_Y, h_Z \rangle$ where:

- $h_C = \{h_{C_n} : C_n \to C'_n\}_{n \in \mathbb{N}}$ is a family of maps;
- $h_O = \{h_{O_n} : O_n \to O'_n\}_{n \in \mathbb{N}}$, is a family of maps preserving $\top$ and $\bot$;
- $h_X : X \to X'$ is a map;
- $h_Y : Y \to Y'$ is a map;
- $h_Z : Z \to Z'$ is a map. ■

It is possible to define $\hat{h}$ the extension of a signature morphism h to formulas, terms, assertions, substitutions, sequents, provisos and rules as follows:

- For each $\varphi \in F(\Sigma)$, $\hat{h}(\varphi) \in F(\Sigma')$ is such that if φ is ξ_i, $\hat{h}(\varphi) = \xi_i$; if φ is $\mathbf{z}$, $\hat{h}(\varphi) = h_Z(\mathbf{z})$; if φ is $c(\varphi_1, \dots, \varphi_k)$, $\hat{h}(\varphi) = h_{C_k}(c)(\hat{h}(\varphi_1), \dots, \hat{h}(\varphi_k))$.
- For each $\theta \in T(\Sigma)$, $\hat{h}(\theta) \in T(\Sigma')$ is such that if θ is τ_i, $\hat{h}(\theta) = \tau_i$; if θ is $\mathbf{x}$, $\hat{h}(\theta) = h_X(\mathbf{x})$; if θ is $\mathbf{y}$, $\hat{h}(\theta) = h_Y(\mathbf{y})$; if θ is $o(\theta_1, \dots, \theta_k)$, $\hat{h}(\theta) = h_{O_k}(o)(\hat{h}(\theta_1), \dots, \hat{h}(\theta_k))$; if θ is $\#\varphi$, $\hat{h}(\theta) = \#\hat{h}(\varphi)$.
- For each $a \in A(\Sigma)$, $\hat{h}(a) \in A(\Sigma')$ is such that if a is $\theta \leq \varphi$ $[\theta \not\leq \varphi]$, $\hat{h}(a) = \hat{h}(\theta) \leq \hat{h}(\varphi)$ $[\hat{h}(a) = \hat{h}(\theta) \not\leq \hat{h}(\varphi)]$; if a is $\varrho(\theta_1, \dots, \theta_k)$ $[\not\varrho(\theta_1, \dots, \theta_k)]$, $\hat{h}(a) = \varrho(\hat{h}(\theta_1), \dots, \hat{h}(\theta_k))$ $[\hat{h}(a) = \not\varrho(\hat{h}(\theta_1), \dots, \hat{h}(\theta_k))]$.

- For each $\rho \in Sbs(A(\Sigma))$, $\hat{h}(\rho) \in Sbs(A(\Sigma'))$ is the map $\hat{h} \circ \rho$.
- For each sequent $s = \langle \Delta_1, \Delta_2 \rangle$ over $A(\Sigma)$, $\hat{h}(s)$ is the sequent $\langle \hat{h}(\Delta_1), \hat{h}(\Delta_2) \rangle$ where $\hat{h}(\Delta_i) \in \mathcal{B}_f(A(\Sigma') \cup \mathcal{G})$ for $i = 1, 2$ is such that for each $\delta \in \Delta_i$ if $\delta \in A(\Sigma)$ then $\hat{h}(\delta) \in A(\Sigma')$ is as defined above and if $\delta \in \mathcal{G}$ then $\hat{h}(\delta) = \delta$.
- For each proviso π over $A(\Sigma)$, $\hat{h}(\pi)$ is the proviso over $A(\Sigma')$ such that $\hat{h}(\pi) : \hat{h}(Sbs(A(\Sigma))) \to \{0, 1\}$.
- For each rule $r = \langle \{s_1, \ldots, s_p\}, s, \pi \rangle$ over $A(\Sigma)$,

$$\hat{h}(r) = \langle \{\hat{h}(s_1), \ldots, \hat{h}(s_p)\}, \hat{h}(s), \hat{h}(\pi) \rangle$$

  is a rule over $A(\Sigma')$.

In the sequel, $\hat{h}$ will be denoted simply by h.

Signatures and signature morphisms constitute a category, named $\mathcal{S}ig$, that has all limits and colimits.

For the purpose of the definition of the map between algebras, the concept of reduct of an algebra *by a signature map* is now introduced.

Definition 4.2.2. Given a Σ'-algebra $\mathbb{A}' = \langle F, T, \cdot_{\mathbb{A}'} \rangle$ and a signature morphism $h : \Sigma \to \Sigma'$, the *reduct of* $\mathbb{A}'$ *by* h is a Σ-algebra $\mathbb{A}'|_h = \langle F, T, \cdot_{\mathbb{A}'|_h} \rangle$ where:

- for each connective $c \in \Sigma$, $c_{\mathbb{A}'|_h} = h(c)_{\mathbb{A}'}$;
- for each operator $o \in \Sigma$, $o_{\mathbb{A}'|_h} = h(o)_{\mathbb{A}'}$;
- $\#_{\mathbb{A}'|_h} = \#_{\mathbb{A}'}$, $\leq_{\mathbb{A}'|_h} = \leq_{\mathbb{A}'}$ and for each $\rho \in \Sigma$, $\varrho_{\mathbb{A}'|_h} = \varrho_{\mathbb{A}'}$. ■

Definition 4.2.3. A *logic system morphism* $m : \mathcal{L} \to \mathcal{L}'$ where $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ and $\mathcal{L}' = \langle \Sigma', \mathcal{R}', \mathcal{A}' \rangle$ is a pair $m = \langle h, a \rangle$ where $h : \Sigma \to \Sigma'$ is a signature morphism such that for each $r = \langle \{s_1, ..., s_p\}, s, \pi \rangle \in \mathcal{R}$ there is a derivation in $\mathcal{R}'$

$$h(s_1), \ldots, h(s_p) \vdash_{\mathcal{C}'} h(s) \lhd h(\pi),$$

and $a : \mathcal{A}' \to \mathcal{A}$ is a map such that $a(\mathbb{A}') = \mathbb{A}'|_h$. ■

Concerning the definition of logic system morphisms, the map between algebras is contravariant in order to guarantee the preservation of entailment: it is essential that each model $\mathbb{A}'$ in $\mathcal{L}'$ have an image by the morphism in $\mathcal{L}$.

Proposition 4.2.4. Logic systems and logic system morphisms constitute a category named $\mathcal{L}og$.

Some important preservation properties hold in the context of $\mathcal{L}og$ as described below. The first one is preservation of derivation.

Proposition 4.2.5. Given $m : \mathcal{L} \to \mathcal{L}'$ if $S \vdash_{\mathcal{C}} s' \lhd \pi$ then $h(S) \vdash_{\mathcal{C}'} h(s') \lhd h(\pi)$.

Proof. The proof follows by induction on the size of a derivation $d = (d_1, \pi_1) \ldots (d_n, \pi_n)$ for $S \vdash_{\mathcal{C}} s' \lhd \pi$.

Base: $d = (d_1, \pi_1)$. So $d_1 = s'$. Then:

- either $d_1 \in S$ and $\pi_1 = \mathbf{up}$ and so $h(d_1) \in h(S)$ and $h(\pi_1) = \mathbf{up}$;
- or there is an assertion that occurs in both sides of d_1 and $\pi_1 = \mathbf{up}$. Then $h(d_1)$ also has an assertion that occurs in both sides of the sequent and $h(\pi_1) = \mathbf{up}$;
- or there are $r \in \mathcal{R}$ and $\rho \in Sbs(A(\Sigma))$ such that $r\rho =$

$$\frac{}{d_1} \lhd \pi_1.$$

  By the definition of m there is a derivation of $h(r)$ in $\mathcal{C}'$. Using the fact that $h(r\rho) = h(r)h(\rho)$ then $\vdash_{\mathcal{C}'} h(d_1) \lhd h(\pi_1)$.

Step: $d = (d_1, \pi_1)(d_2, \pi_2) \ldots (d_n, \pi_n)$ is a derivation in $\mathcal{L}$. Then:

- either $d_1 \in S$ and $\pi_1 = \mathbf{up}$ and so $h(d_1) \in h(S)$ and $h(\pi_1) = \mathbf{up}$;
- or there is an assertion that occurs in both sides of d_1 and $\pi_1 = \mathbf{up}$. Then $h(d_1)$ also has an assertion that occurs in both sides of the sequent and $h(\pi_1) = \mathbf{up}$;
- or there are $r = \langle \{s_1, \ldots, s_q\}, s, \pi' \rangle \in \mathcal{R}, \rho \in Sbs(A(\Sigma)), q \in \mathbb{N}$ and $i_1, ..., i_q \in \{2, 3, ..., n\}$ such that $r\rho =$

$$\frac{d_{i_1} \quad \ldots \quad d_{i_q}}{d_1} \lhd \pi$$

  and $\pi_1 = \pi \cap \pi_{i_1} \cap \ldots \cap \pi_{i_q}$. By the definition of m there is a derivation $d_r = (d_{r_1}, \pi_{r_1}) \ldots (d_{r_m}, \pi_{r_m})$ showing $h(s_1), \ldots, h(s_q) \vdash_{\mathcal{C}'} h(s) \lhd h(\pi')$. Observe that for $j = 1, \ldots, q$, $h(\rho)(h(s_j)) = h(s_j\rho) = h(d_{i_j})$ and $h(\rho)(h(s)) = h(s\rho) = h(d_1)$. By the induction hypothesis, $(d_2, \pi_2) \ldots (d_n, \pi_n)$ has an "equivalent" derivation that will be denoted as d_{Step}. So the final derivation in $\mathcal{C}'$ is $h(\rho)(d_r)d_{Step}$ where each justification 'hyp' in d_r is replaced by the correspondent reference of $h(d_{i_j})$ in d_{Step}.

□

Given a morphism between logic systems and assignments over the destination algebra, it is possible to consider corresponding assignments over the source algebra in order to establish interesting preservation properties.

Given a logic system morphism $m = \langle h, a\rangle$ and an algebra $\mathbb{A}' \in \mathcal{A}'$ it is denoted by α and β the assignments over $\mathbb{A} = a(\mathbb{A}')$ induced by the assignments α' and β' over $\mathbb{A}'$ such that the diagrams in Figure 4.2 commute.

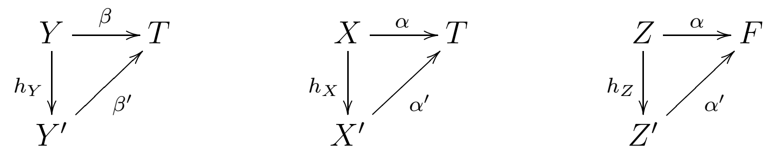

Figure 4.2: Commutative diagrams for induced assignments

The following lemma establishes the preservation of the denotation of a formula by a logic system morphism.

Lemma 4.2.6. Let φ be a formula, $\langle h, a\rangle : \mathcal{L} \to \mathcal{L}'$ a logic system morphism, $\mathbb{A}' \in \mathcal{A}'$, α' an assignment over $\mathbb{A}'$, and α be $\alpha' \circ h$. Then $[\![\varphi]\!]_{a(\mathbb{A}')\alpha} = [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'}$.

Proof. The proof follows by induction on the structure of φ.

- φ is $\mathbf{z}$. Then:
  $[\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha} = \alpha(\mathbf{z}) = \alpha'(h_Z(\mathbf{z})) = [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}$.
- φ is $c(\varphi_1, \ldots, \varphi_k)$. Then:
  $[\![c(\varphi_1, \ldots, \varphi_k)]\!]_{a(\mathbb{A}')\alpha} = c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha}, \ldots, [\![\varphi_k]\!]_{a(\mathbb{A}')\alpha}) = h(c)_{\mathbb{A}'}([\![h(\varphi_1)]\!]_{\mathbb{A}'\alpha'}, \ldots, [\![h(\varphi_k)]\!]_{\mathbb{A}'\alpha'}) = [\![h(c(\varphi_1, \ldots, \varphi_k))]\!]_{\mathbb{A}'\alpha'}$.

□

The next two lemmas show the preservation of the denotations of the relations used in assertions. They are used in the proof of Proposition 4.2.9.

Lemma 4.2.7. Given $\mathbb{A}' \in \mathcal{A}'$ and assignments α' and β' over $\mathbb{A}'$,

$$\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\varphi]\!]_{a(\mathbb{A}')\alpha}\rangle \in \leq_{a(\mathbb{A}')} \text{ iff } \langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'},$$

where α and β are as indicated before.

Proof. The proof follows by structural induction on θ and φ.

- φ is $\mathbf{z} \in Z$.

  - θ is $\mathbf{x} \in X$. Then:

  $\langle [\![\mathbf{x}]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha} \rangle \in \leq_{a(\mathbb{A}')}$

$$\begin{array}{ll} \text{iff} & \langle \alpha(\mathbf{x}), \alpha(\mathbf{z}) \rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle \alpha'(h_X(\mathbf{x})), \alpha'(h_Z(\mathbf{z})) \rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \rangle \in \leq_{\mathbb{A}'} . \end{array}$$

  - θ is $\mathbf{y} \in Y$. Then:

  $\langle [\![\mathbf{y}]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha} \rangle \in \leq_{a(\mathbb{A}')}$

$$\begin{array}{ll} \text{iff} & \langle \beta(\mathbf{y}), \alpha(\mathbf{z}) \rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle \beta'(h_Y(\mathbf{y})), \alpha'(h_Z(\mathbf{z})) \rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(\mathbf{y})]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \rangle \in \leq_{\mathbb{A}'} . \end{array}$$

  - θ is $o(\theta_1, \ldots, \theta_k)$ with $o \in O_k$ and $\theta_1, \ldots, \theta_k \in T(\Sigma)$. Then:

  $\langle [\![o(\theta_1, \ldots, \theta_k)]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha} \rangle \in \leq_{a(\mathbb{A}')}$

$$\begin{array}{ll} \text{iff} & \langle o_{a(\mathbb{A}')}([\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta}, \ldots, [\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta}), \alpha(\mathbf{z}) \rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle h(o)_{\mathbb{A}'}([\![h(\theta_1)]\!]_{\mathbb{A}'\alpha'\beta'}, \ldots, [\![h(\theta_k)]\!]_{\mathbb{A}'\alpha'\beta'}), \alpha'(h_z(\mathbf{z})) \rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(o)(h(\theta_1), \ldots, h(\theta_k))]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(o(\theta_1, \ldots, \theta_k))]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \rangle \in \leq_{\mathbb{A}'} . \end{array}$$

  - θ is $\#\psi$ with ψ being $\mathbf{z} \in Z$. Then:

  $\langle [\![\#\mathbf{z}]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha} \rangle \in \leq_{a(\mathbb{A}')}$

$$\begin{array}{ll} \text{iff} & \langle \#_{a(\mathbb{A}')}([\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha\beta}), \alpha(\mathbf{z}) \rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle \#_{a(\mathbb{A}')}(\alpha(\mathbf{z})), \alpha(\mathbf{z}) \rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle \#_{\mathbb{A}'}(\alpha'(h_Z(\mathbf{z}))), \alpha'(h_Z(\mathbf{z})) \rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle \#_{\mathbb{A}'}([\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}), [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![\#h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(\#\mathbf{z})]\!]_{\mathbb{A}'\alpha'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \rangle \in \leq_{\mathbb{A}'} . \end{array}$$

  - θ is $\#\psi$ with ψ being $c(\psi_1, \ldots, \psi_k)$ where $c \in C_k$ and $\psi_1, \ldots, \psi_k \in F(\Sigma)$. Then:

$$\langle [\![\#c(\psi_1,\ldots,\psi_k)]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha}\rangle \in \leq_{a(\mathbb{A}')}$$

$$\begin{array}{ll} \text{iff} & \langle \#_{a(\mathbb{A}')}([\![c(\psi_1,\ldots,\psi_k)]\!]_{a(\mathbb{A}')\alpha}), \alpha(\mathbf{z})\rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle \#_{a(\mathbb{A}')}(c_{a(\mathbb{A}')}([\![\psi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\psi_k]\!]_{a(\mathbb{A}')\alpha})), \alpha(\mathbf{z})\rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle \#_{\mathbb{A}'}(h(c)_{\mathbb{A}'}([\![h(\psi_1)]\!]_{\mathbb{A}'\alpha'},\ldots,[\![h(\psi_k)]\!]_{\mathbb{A}'\alpha'})), \alpha'(h_Z(\mathbf{z}))\rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle \#_{\mathbb{A}'}([\![h(c)(h(\psi_1),\ldots,h(\psi_k))]\!]_{\mathbb{A}'\alpha'}), [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![\#h(c(\psi_1,\ldots,\psi_k))]\!]_{\mathbb{A}'\alpha'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(\#c(\psi_1,\ldots,\psi_k))]\!]_{\mathbb{A}'\alpha'}, [\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'} . \end{array}$$

- φ is $c(\varphi_1,\ldots,\varphi_k)$ with $c \in C_k$ and $\varphi_1,\ldots,\varphi_k \in F(\Sigma)$.

It is showed only the case where θ is $\mathbf{x} \in X$, all other cases are similar to the cases presented above.

$$\langle [\![\mathbf{x}]\!]_{a(\mathbb{A}')\alpha\beta}, [\![c(\varphi_1,\ldots,\varphi_k)]\!]_{a(\mathbb{A}')\alpha}\rangle \in \leq_{a(\mathbb{A}')}$$

$$\begin{array}{ll} \text{iff} & \langle \alpha(\mathbf{x}), c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha},\ldots,[\![\varphi_k]\!]_{a(\mathbb{A}')\alpha})\rangle \in \leq_{a(\mathbb{A}')} \\ \text{iff} & \langle \alpha'(h_X(\mathbf{x})), h(c)_{\mathbb{A}'}([\![h(\varphi_1)]\!]_{\mathbb{A}'\alpha'},\ldots,[\![h(\varphi_k)]\!]_{\mathbb{A}'\alpha'})\rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(c)(h(\varphi_1),\ldots,h(\varphi_k))]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'} \\ \text{iff} & \langle [\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(c(\varphi_1,\ldots,\varphi_k))]\!]_{\mathbb{A}'\alpha'}\rangle \in \leq_{\mathbb{A}'} . \end{array}$$

□

The lemma that follows establishes the preservation of satisfaction for the relations $\varrho \in \mathbb{P}$.

Lemma 4.2.8. Given $\mathbb{A}' \in \mathcal{A}'$ and assignments α' and β' over $\mathbb{A}'$,

$$\langle [\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta},\ldots,[\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \varrho_{a(\mathbb{A}')} \text{ iff } \langle [\![h(\theta_1)]\!]_{\mathbb{A}'\alpha'\beta'},\ldots,[\![h(\theta_k)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \varrho_{\mathbb{A}'},$$

where α and β are as in Figure 4.2.

Proof. The proof follows by structural induction on each θ_i for $i = 1,\ldots,k$. Without loss of generality, only the case where $\varrho \in \mathbb{P}_1$ is shown. The proof for $\mathbb{P}_n$ with $n \geq 2$ is similar.

- θ_1 is $\mathbf{x} \in X$.

  $[\![\mathbf{x}]\!]_{a(\mathbb{A}')\alpha\beta} \in \varrho_{a(\mathbb{A}')}$ iff $\alpha(\mathbf{x}) \in \varrho_{a(\mathbb{A}')}$ iff $\alpha'(h_X(\mathbf{x})) \in \varrho_{\mathbb{A}'}$ iff $[\![h(\mathbf{x})]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'}$.

- θ_1 is $\mathbf{y} \in Y$.

  $[\![\mathbf{y}]\!]_{a(\mathbb{A}')\alpha\beta} \in \varrho_{a(\mathbb{A}')}$ iff $\beta(\mathbf{y}) \in \varrho_{a(\mathbb{A}')}$ iff $\beta'(h_Y(\mathbf{y})) \in \varrho_{\mathbb{A}'}$ iff $[\![h(\mathbf{y})]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'}$.

- θ_1 is $o(\theta'_1, \ldots, \theta'_l)$ with $o \in O_l$ and $\theta'_1, \ldots, \theta_l \in T(\Sigma)$.

  $[\![o(\theta'_1, \ldots, \theta'_l)]\!]_{a(\mathbb{A}'\alpha\beta)} \in \varrho_{a(\mathbb{A}')}$

$$\begin{array}{ll} \text{iff} & o_{a(\mathbb{A}')}([\![\theta'_1]\!]_{a(\mathbb{A}')\alpha\beta}, \ldots, [\![\theta'_l]\!]_{a(\mathbb{A}')\alpha\beta}) \in \varrho_{a(\mathbb{A}')} \\ \text{iff} & h(o)_{\mathbb{A}'}([\![h(\theta'_1)]\!]_{\mathbb{A}'\alpha'\beta'}, \ldots, [\![h(\theta'_l)]\!]_{\mathbb{A}'\alpha'\beta'}) \in \varrho_{\mathbb{A}'} \\ \text{iff} & [\![h(o)(h(\theta'_1), \ldots, h(\theta'_l))]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'} \\ \text{iff} & [\![h(o(\theta'_1, \ldots, \theta'_l))]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'}. \end{array}$$

- θ_1 is $\#\varphi$ where φ is $\mathbf{z} \in Z$.

  $[\![\#\mathbf{z}]\!]_{a(\mathbb{A}')\alpha\beta} \in \varrho_{a(\mathbb{A}')}$

$$\begin{array}{ll} \text{iff} & \#_{a(\mathbb{A}')}[\![\mathbf{z}]\!]_{a(\mathbb{A}')\alpha\beta} \in \varrho_{a(\mathbb{A}')} \\ \text{iff} & \#_{a(\mathbb{A}')}\alpha(\mathbf{z}) \in \varrho_{a(\mathbb{A}')} \\ \text{iff} & \#_{\mathbb{A}'}\alpha'(h_Z(\mathbf{z})) \in \varrho_{\mathbb{A}'} \\ \text{iff} & \#_{\mathbb{A}'}[\![h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'} \in \varrho_{\mathbb{A}'} \\ \text{iff} & [\![\#h(\mathbf{z})]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'} \\ \text{iff} & [\![h(\#\mathbf{z})]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'}. \end{array}$$

- θ_1 is $\#\varphi$ where φ is $c(\varphi_1, \ldots, \varphi_l)$, $c \in C_l$ and $\varphi_1, \ldots, \varphi_l \in F(\Sigma)$.

  $[\![\#c(\varphi_1, \ldots, \varphi_l)]\!]_{a(\mathbb{A}')\alpha\beta} \in \varrho_{a(\mathbb{A}')}$

$$\begin{array}{ll} \text{iff} & \#_{a(\mathbb{A}')}[\![c(\varphi_1, \ldots, \varphi_l)]\!]_{a(\mathbb{A}')\alpha} \in \varrho_{a(\mathbb{A}')} \\ \text{iff} & \#_{a(\mathbb{A}')}c_{a(\mathbb{A}')}([\![\varphi_1]\!]_{a(\mathbb{A}')\alpha}, \ldots, [\![\varphi_l]\!]_{a(\mathbb{A}')\alpha}) \in \varrho_{a(\mathbb{A}')} \\ \text{iff} & \#_{\mathbb{A}'}h(c)_{\mathbb{A}'}([\![h(\varphi_1)]\!]_{\mathbb{A}'\alpha'}, \ldots, [\![h(\varphi_l)]\!]_{\mathbb{A}'\alpha}) \in \varrho_{\mathbb{A}'} \\ \text{iff} & \#_{\mathbb{A}'}[\![h(c)(h(\varphi_1), \ldots, h(\varphi_l))]\!]_{\mathbb{A}'\alpha'} \in \varrho_{\mathbb{A}'} \\ \text{iff} & [\![\#h(c(\varphi_1, \ldots, \varphi_l))]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'} \\ \text{iff} & [\![h(\#c(\varphi_1, \ldots, \varphi_l))]\!]_{\mathbb{A}'\alpha'\beta'} \in \varrho_{\mathbb{A}'}. \end{array}$$

□

The next proposition shows that logic system morphisms preserve satisfaction of assertions.

Proposition 4.2.9. Given $m : \mathcal{L} \to \mathcal{L}'$, $\mathbb{A}' \in \mathcal{A}'$, and α' and β' assignments over $\mathbb{A}'$,

$$a(\mathbb{A}')\alpha\beta \Vdash_{\mathcal{L}} \gamma \text{ iff } \mathbb{A}'\alpha'\beta' \Vdash_{\mathcal{L}'} h(\gamma)$$

where α and β are as in Figure 4.2 and γ is either an assertion or a sequent.

Proof. Suppose γ is an assertion, the proof follows by case analysis.

(i). $a(\mathbb{A}')\alpha\beta \Vdash \theta \leq \varphi$ iff $\langle[\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\varphi]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \leq_{a(\mathbb{A}')}$ iff (by Lemma 4.2.7) $\langle[\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \leq_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta) \leq h(\varphi)$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta \leq \varphi)$.

(ii). $a(\mathbb{A}')\alpha\beta \Vdash \theta \not\leq \varphi$ iff $\langle [\![\theta]\!]_{a(\mathbb{A}')\alpha\beta}, [\![\varphi]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \notin \leq_{a(\mathbb{A}')}$ iff (by Lemma 4.2.7) $\langle [\![h(\theta)]\!]_{\mathbb{A}'\alpha'\beta'}, [\![h(\varphi)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \notin \leq_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta) \not\leq h(\varphi)$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\theta \not\leq \varphi)$.

(iii). $a(\mathbb{A}')\alpha\beta \Vdash \varrho(\theta_1, \ldots, \theta_k)$ iff $\langle [\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta}, \ldots, [\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \in \varrho_{a(\mathbb{A}')}$ iff (by Lemma 4.2.8) $\langle [\![h(\theta_1)]\!]_{\mathbb{A}'\alpha'\beta'}, \ldots, [\![h(\theta_k)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \in \varrho_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash \varrho(h(\theta_1), \ldots, h(\theta_k))$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\varrho(\theta_1, \ldots, \theta_k))$.

(iv). $a(\mathbb{A}')\alpha\beta \Vdash \not\varrho(\theta_1, \ldots, \theta_k)$ iff $\langle [\![\theta_1]\!]_{a(\mathbb{A}')\alpha\beta}, \ldots, [\![\theta_k]\!]_{a(\mathbb{A}')\alpha\beta}\rangle \notin \varrho_{a(\mathbb{A}')}$ iff (by Lemma 4.2.8) $\langle [\![h(\theta_1)]\!]_{\mathbb{A}'\alpha'\beta'}, \ldots, [\![h(\theta_k)]\!]_{\mathbb{A}'\alpha'\beta'}\rangle \notin \varrho_{\mathbb{A}'}$ iff $\mathbb{A}'\alpha'\beta' \Vdash \not\varrho(h(\theta_1), \ldots, h(\theta_k))$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\not\varrho(\theta_1, \ldots, \theta_k))$.

Suppose now that γ is a sequent $\Delta' \to \Delta''$. Then $a(\mathbb{A}')\alpha\beta \Vdash \Delta' \to \Delta''$ iff $a(\mathbb{A}')\alpha\beta \Vdash \delta$ for some $\delta \in \Delta'' \cup \overline{\Delta'}$ iff by (i)–(iv) above $\mathbb{A}'\alpha'\beta' \Vdash h(\delta)$ with $h(\delta) \in h(\Delta'' \cup \overline{\Delta'})$ iff $\mathbb{A}'\alpha'\beta' \Vdash h(\Delta' \to \Delta'')$. □

The following result states that soundness of rules is preserved by logic systems morphism.

Proposition 4.2.10. Given a sound logic system $\mathcal{L}$ and a logic system morphism $m : \mathcal{L} \to \mathcal{L}'$ where $m = \langle m_h, m_a\rangle$ with m_h injective, if $r = \langle\{s_1, \ldots, s_n\}, s, \pi\rangle \in \mathcal{R}$ then $m_h(s_1), \ldots, m_h(s_n) \vDash_{\mathcal{L}'} m_h(s) \lhd m_h(\pi)$.

Proof. Let $\mathbb{A}' \in \mathcal{A}'$, α' an unbound assignment over $\mathbb{A}'$ and ρ a ground substitution over $m_h(A(\Sigma))$ such that $m_h(\pi)(\rho) = 1$. Suppose

$$\mathbb{A}'\alpha' \Vdash m_h(s_i)\rho \text{ for all } i = 1, \ldots, n.$$

Since m_h is injective, m_h^{-1} is a substitution over $A(\Sigma)$ such that $\pi(m_h^{-1}(\rho)) = 1$. So, by Proposition 4.2.9,

$$a(\mathbb{A}')\alpha \Vdash s_i m_h^{-1}(\rho) \text{ for all } i = 1, \ldots, n$$

with $\alpha = \alpha' \circ m_h$. But $r = \langle\{s_1, \ldots, s_n\}, s, \pi\rangle \in \mathcal{R}$ and since $\mathcal{L}$ is sound

$$s_1, \ldots, s_n \vDash_{\mathcal{L}} s \lhd \pi.$$

By definition of entailment, if $\mathbb{A}\alpha^* \Vdash s_i\sigma$ for $i = 1, \ldots, n$, with σ any ground substitution over $A(\Sigma)$ such that $\pi(\sigma) = 1$ and any unbound assignment α^* over $\mathbb{A}$ then $\mathbb{A}\alpha^* \Vdash s\sigma$. So in particular for $a(\mathbb{A}')$, α and $m_h^{-1}(\rho)$,

$$a(\mathbb{A}')\alpha \Vdash s\ m_h^{-1}(\rho).$$

Again by Proposition 4.2.9 $\mathbb{A}'\alpha' \Vdash m_h(s)\rho$. Thus

$$m_h(s_1), \ldots, m_h(s_n) \vDash_{\mathcal{L}'} m_h(s) \lhd m_h(\pi).$$

□

4.2.2 Categorical fusion

In this subsection, the concept of fusion is generalized from (normal) modal logic to relational LSLTV as introduced in Definition 2.5.3. A categorical approach is adopted to attain this goal.

The categorical fusion is given by the next definition: it is the pushout in $\mathcal{L}og$ of the diagram of Figure 4.3 where $\mathcal{L}_C$ is a subsystem of $\mathcal{L}_1$ and $\mathcal{L}_2$ in what concerns the signature and the set of rules.

Definition 4.2.11. Let $\mathcal{L}_1 = \langle \Sigma_1, \mathcal{R}_1, \mathcal{A}_1 \rangle$ and $\mathcal{L}_2 = \langle \Sigma_2, \mathcal{R}_2, \mathcal{A}_2 \rangle$ be two relational LSLTV. The *fusion by sharing* $\mathcal{L}_C$ *of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ is the object of the pushout in $\mathcal{L}og$ of the diagram of Figure 4.3, where $\mathcal{L}_C = \langle \Sigma_C, \mathcal{R}_C, \mathcal{A}_C \rangle$ is a relational LSLTV such that:

- $\Sigma_C \subseteq \Sigma_i$ for $i = 1, 2$,
- $\mathcal{R}_C \subseteq \mathcal{R}_i$ for $i = 1, 2$,
- $\mathcal{A}_C$ is the class of all Σ_C-algebras,

and the morphisms $inc_i = \langle (inc_i)_h, (inc_i)_a \rangle$ are inclusions where $(inc_i)_a(\mathbb{A}_i) = \mathbb{A}_i|_{(inc_i)_h}$ for $i = 1, 2$. ■

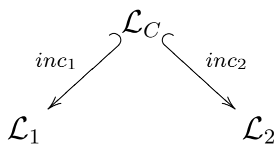

Figure 4.3: Diagram for fusion

It is worthwhile to observe that the algebraic fusion of a given pair $\mathcal{L}_1$ and $\mathcal{L}_2$ of (normal) modal logics, mentioned in Section 4.1, coincides with the object of the pushout described in Definition 4.2.11 when $\mathcal{L}_C$ is such that:

- Σ_C is the signature Σ_P together with $\mathbf{I} \in O_1$ and $\mathbf{lb} \in O_2$;
- $\mathcal{R}_C$ is the set constituted by the rules **I**, Ω**I**, **lb**1, **lb**2 and the rules in Tables 2.1, 2.2 and 2.3;
- $\mathcal{A}_C$ is the class of all Σ_C-algebras.

It is now provided a more detailed characterization of the colimit of the diagram of Figure 4.3, that is, a description of the components of the object and of the morphisms that constitute the pushout. It should be stressed that the object of the pushout is the fusion of the logic systems.

Proposition 4.2.12. The triple $\langle \mathcal{L}, m_1, m_2 \rangle$ where $\mathcal{L} = \langle \Sigma, \mathcal{R}, \mathcal{A} \rangle$ and $m_i = \langle (m_i)_h, (m_i)_a \rangle : \mathcal{L}_i \to \mathcal{L}$ for $i = 1, 2$, are such that

- $\langle \Sigma, (m_1)_h, (m_2)_h \rangle$ is the pushout in $\mathcal{S}ig$ of the morphisms $(inc_1)_h$ and $(inc_2)_h$ introduced in Definition 4.2.11*;*
- $\mathcal{R} = (m_1)_h \circ (inc_1)_h(\mathcal{R}_C) \cup (m_1)_h(\mathcal{R}_1 \backslash \mathcal{R}_C) \cup (m_2)_h(\mathcal{R}_2 \backslash \mathcal{R}_C)$*;*
- $\mathcal{A} = \{\mathbb{A}_1\_\mathbb{A}_2 = \langle F, T, \cdot_{\mathbb{A}_1\_\mathbb{A}_2} \rangle : (inc_1)_a(\mathbb{A}_1) = (inc_2)_a(\mathbb{A}_2) = \langle F, T, \cdot_{\mathbb{A}_C} \rangle$ and $(m_1)_h(c)_{\mathbb{A}_1\_\mathbb{A}_2} = c_{\mathbb{A}_1}$ for $c \in C^1, (m_2)_h(c)_{\mathbb{A}_1\_\mathbb{A}_2} = c_{\mathbb{A}_2}$ for $c \in C^2, (m_1)_h(o)_{\mathbb{A}_1\_\mathbb{A}_2} = o_{\mathbb{A}_1}$ for $o \in O^1, (m_2)_h(o)_{\mathbb{A}_1\_\mathbb{A}_2} = o_{\mathbb{A}_2}$ for $o \in O^2, \#_{\mathbb{A}_1\_\mathbb{A}_2} = \#_{\mathbb{A}_1}, \leq_{\mathbb{A}_1\_\mathbb{A}_2} = \leq_{\mathbb{A}_1}, \varrho_{\mathbb{A}_1\_\mathbb{A}_2} = \varrho_{\mathbb{A}_1}\}$ where C^i and O^i are, respectively, the set of connectives and the set of operators of Σ_i for $i = 1, 2$;
- $(m_i)_a(\mathbb{A}_1\_\mathbb{A}_2) = \mathbb{A}_i$ for $i = 1, 2$

is the pushout of the diagram of Figure 4.3. So, $\mathcal{L}$ is the fusion of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof. Suppose there is $\langle \mathcal{L}', l_1, l_2 \rangle$ such that $l_1 : \mathcal{L}_1 \to \mathcal{L}'$, $l_2 : \mathcal{L}_2 \to \mathcal{L}'$ and $l_1 \circ inc_1 = l_2 \circ inc_2$. It has to be shown that there is a unique morphism $u : \mathcal{L} \to \mathcal{L}'$ such that $u \circ m_1 = l_1$ and $u \circ m_2 = l_2$. Let $u_a(\mathbb{A}') = ((l_1)_a(\mathbb{A}'))\_((l_2)_a(\mathbb{A}'))$. Unicity of u: Suppose there is $n : \mathcal{L} \to \mathcal{L}'$ such that $n \circ m_1 = l_1$ and $n \circ m_2 = l_2$. Then $n_h = u_h$ since $\langle \Sigma, (m_1)_h, (m_2)_h \rangle$ is the pushout in $\mathcal{S}ig$. Let $\mathbb{A}' \in \mathcal{A}'$ and $n_a(\mathbb{A}') = \mathbb{A}_x\_\mathbb{A}_y$. So, $(m_1)_a \circ n_a(\mathbb{A}') = (m_1)_a(\mathbb{A}_x\_\mathbb{A}_y) = \mathbb{A}_x = (l_1)_a(\mathbb{A}')$ and $(m_2)_a \circ n_a(\mathbb{A}') = (m_2)_a(\mathbb{A}_x\_\mathbb{A}_y) = \mathbb{A}_y = (l_2)_a(\mathbb{A}')$. Thus, $n_a = u_a$. Therefore, $n = u$. ☐

An important point about the categorical account of fusion is that it is possible to change the shared connectives and operators by simply changing the "common logic system", $\mathcal{L}_C$, of Definition 4.2.11. This contrasts with the usual definition of fusion, where the shared connectives are fixed, being the propositional connectives. On the other hand, the definition of categorical fusion assumes at least that $\mathcal{L}_C$ is the common core of $\mathcal{L}_1$ and $\mathcal{L}_2$. So it is possible to consider extensions of that logic. For instance, if $\mathcal{L}_C$ is system K then the fusion will shared the modality as well.

4.3 Soundness and Completeness Preservation in Fusion

This section shows that soundness and completeness are preserved by the categorical fusion of Definition 4.2.11. While preservation of soundness is immediate, for preservation of completeness it has to be assumed that both component systems have rules endowed with persistent provisos.

Theorem 4.3.1. The fusion of sound relational LSLTV is sound.

Proof. Consider the logic systems $\mathcal{L}_1$ and $\mathcal{L}_2$ and their fusion $\mathcal{L}$ using a logic system $\mathcal{L}_C$ for specifying the shared connectives as described in Definition 4.2.11.

The proof follows by applying Proposition 4.2.10 to each rule $r = \langle\{s_1, \dots, s_n\}, s, \pi\rangle$ of $\mathcal{R}$. Several cases have to be considered:

- $r \in (m_1)_h(\mathcal{R}_1 \backslash \mathcal{R}_C)$, that is, it is a rule from $\mathcal{L}_1$. As $\mathcal{L}_1$ is sound, by Proposition 4.2.10 $(m_1)_h(s_1), \dots, (m_1)_h(s_n) \vDash_{\mathcal{L}} (m_1)_h(s) \lhd (m_1)_h(\pi)$. The reasoning is analogous if $r \in (m_2)_h(\mathcal{R}_2 \backslash \mathcal{R}_C)$;
- $r \in \mathcal{R}_C$. Then $r = (m_1)_h(r') = (m_2)_h(r'')$ for some $r' \in \mathcal{R}_1$ and $r'' \in \mathcal{R}_2$, so it is the same case as before.

Therefore, each rule in $\mathcal{R}$ is entailed in $\mathcal{L}$. So the class of algebras $\mathcal{A}$ is appropriate for $\langle\Sigma, \mathcal{R}\rangle$. $\square$

Using the result for completeness given in Theorem 2.5.12, it is possible to show the preservation of completeness by the fusion under mild conditions.

Theorem 4.3.2. The fusion of two full relational LSLTV with rules endowed with persistent provisos is complete.

Proof. From Theorem 2.5.12 it can be concluded that if a logic system is full with rules endowed with persistent provisos, then it is complete.

Let $\mathcal{L}_i = \langle\Sigma_i, \mathcal{R}_i, \mathcal{A}_i\rangle$ with $i = 1, 2$ be the component logic systems and $\mathcal{L} = \langle\Sigma, \mathcal{R}, \mathcal{A}\rangle$ their fusion.

As the rules of $\mathcal{R}$ come only from $\mathcal{R}_1$ or $\mathcal{R}_2$ and both have only rules endowed with persistent provisos, so it will have $\mathcal{R}$.

Taking an appropriate Σ-algebra $\mathbb{A}$ for the rules in $\mathcal{R}$ it must be shown that $\mathbb{A} \in \mathcal{A}$.

Observe that $\mathbb{A}|_{\Sigma_1}$ is appropriate for the calculus $\langle \Sigma_1, \mathcal{R}_1 \rangle$ since all rules $r = \langle \{s_1, \ldots, s_p\},\ s, \pi \rangle$ that belongs to $\mathcal{R}_1$ are also in $\mathcal{R}$ and since if $s_1, \ldots, s_p \vDash_{\mathbb{A}} s \lhd \pi$ then $s_1, \ldots, s_p \vDash_{\mathbb{A}|_{\Sigma_1}} s \lhd \pi$ (notice that $s_1, \ldots, s_p, s$ and π are over $A(\Sigma_1)$). So $\mathbb{A}|_{\Sigma_1} \in \mathcal{A}_1$. The same holds for $\mathbb{A}|_{\Sigma_2}$. So $\mathbb{A} = \mathbb{A}|_{\Sigma_1} \_ \mathbb{A}|_{\Sigma_2} \in \mathcal{A}$. $\square$

Chapter 5

Combined Logic Systems

This chapter is dedicated to illustrate the results established in the preceding chapter. Several combinations of LSLTV are investigated and preservation of soundness and completeness are analysed. A special emphasis is put in the analysis of the completeness of the fusion of the non-normal partition logic (introduced in [FF05]) with itself. In [FF05] this specific combination was the counter-example to the strategy of defining fusion of non-normal modal logics as successive modalizations.

5.1 Systems Obtained by Fusion

This section starts by illustrating the algebraic account of fusion. It is described the combination of the standard deontic logic system introduced in Example 3.1.9 and the Löb provability logic introduced in Example 3.1.10.

Example 5.1.1. The fusion of the logic system SDL presented in Example 3.1.9 and the logic system $\mathcal{GL}$ presented in Example 3.1.10 is such that:

- the signature is Σ as described in Definition 4.1.1, where, for simplicity, $\Box_D$ is used instead of $\Box'$ (the modality of SDL) and $\Box_{GL}$ for $\Box''$ (the modality of $\mathcal{GL}$). Note that expressions like $\Box_D \Box_{GL} \varphi$ have the intended meaning that it is obligatory that φ is provable in a certain theory $\mathbf{T}$;
- the set of rules is as described in Definition 4.1.1 (actually, it is $\mathcal{R}_D \cup \mathcal{R}_{GL}$ with the appropriate substitutions of $\Box$ and $\mathbf{N}$);
- the class $\mathcal{A}$ is constituted by the algebras induced by generalized Kripke structures (see Definition 3.1.5) with two relations such that one accessibility rela-

tion is right-unbounded and the other accessibility relation is transitive and right linear. ∎

An example of a deduction in the context of the sequent calculus resulting from the fusion of SDL and $\mathcal{GL}$ is now presented.

Example 5.1.2. It is now presented a deduction for

$$\Box_D(\Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$$

in the context of the logic system resulting of the fusion of SDL and $\mathcal{GL}$, as described in Example 5.1.1:

1.		$\to$	$\top \leq \Box_D(\Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$	R$\Box_D$ 2
2.		$\to$	$\mathbf{N}_D(\top) \leq \Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$	transF3,4
3.		$\to$	$\mathbf{N}_D(\top) \sqsubseteq \top$	$\top$
4.		$\to$	$\top \leq \Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}\xi_2))$	RgenF 5
5.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top$	$\to$	$\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2) \supset (\Box_{GL}\xi_1 \supset \Box_{GL}1\xi_2)$	R$\supset$ 6
6.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top,$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$	$\to$	$\mathbf{y}_1 \leq \Box_{GL}\xi_1 \supset \Box_{GL}\xi_2$	R$\supset$ 7
7.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_1 \leq \Box_{GL}\xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$	$\to$	$\mathbf{y}_1 \leq \Box_{GL}\xi_2$	R$\Box_{GL}$ 8
8.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_1 \leq \Box_{GL}\xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$	$\to$	$\mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_2$	RgenF 9
9.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_1 \leq \Box_{GL}\xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\to$	$\mathbf{y}_2 \leq \xi_2$	L$\Box_{GL}$ 10
10.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\to$	$\mathbf{y}_2 \leq \xi_2$	LgenF 11,12,13
11.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\to$	$\mathbf{y}_2 \leq \xi_2$	L$\Box_{GL}$14
12.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top$ $\mathbf{y}_1 \leq \Box_{GL}(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\to$	$\mathbf{y}_2 \leq \xi_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	ax
13.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1$ $\mathbf{y}_1 \leq K_1(\xi_1 \supset \xi_2)$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\to$	$\mathbf{y}_2 \leq \xi_2, \Omega\mathbf{y}_2$	ax
14.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1 \supset \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(y_1), \Omega\mathbf{y}_2$	$\to$	$\mathbf{y}_2 \leq \xi_2$	LgenF 15,16,17
15.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\mathbf{N}_{GL}(\mathbf{y}_1) \leq \xi_1 \supset \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\to$	$\mathbf{y}_2 \leq \xi_2, \Omega\mathbf{y}_2$	ax

16.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$ $\Omega\mathbf{y}_2, \Omega\mathbf{y}_2$	$\to$	$\mathbf{y}_2 \leq \xi_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	ax
17.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \leq \xi_1 \supset \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\to$	$\mathbf{y}_2 \leq \xi_2$	L$\supset$18,19
18.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \Omega\mathbf{y}_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1)$	$\to$	$\mathbf{y}_2 \leq \xi_2, \mathbf{y}_2 \leq \xi_1$	ax
19.	$\Omega\mathbf{y}_1, \mathbf{y}_1 \sqsubseteq \top, \mathbf{y}_2 \leq \xi_1$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \leq \xi_2$ $\Omega\mathbf{y}_2, \mathbf{y}_2 \sqsubseteq \mathbf{N}_{GL}(\mathbf{y}_1), \Omega\mathbf{y}_2$	$\to$	$\mathbf{y}_2 \leq \xi_2$	ax ■

The first example of categorical fusion, Example 5.1.3 illustrates the combination of two normal modal systems: one is the standard deontic logic and the other is used for knowledge logic. This system reasons about knowledge and obligation and may be used, for instance, in multi-agent systems in computer science since it allows to talk about obligatory behaviours and knowledge of the agents.

Example 5.1.3. The fusion of the standard deontic modal system SDL of Example 3.1.9 with the knowledge modal system $\mathcal{K}$ of Example 3.1.11 sharing $\mathcal{L}_C$ where:

- Σ_C coincides with Σ_P where Σ_P is the signature of propositional logic;
- $\mathcal{R}_C$ is the set with the structural rules of Table 2.1, the order rules of Table 2.2 and the rules for the classical connectives presented in Table 2.3;
- $\mathcal{A}_C$ is the class of all Σ_C-algebras;

is obtained by taking the morphisms $inc_1 : \mathcal{L}_C \to \mathsf{SDL}$ and $inc_2 : \mathcal{L}_C \to \mathcal{K}$, where $inc_1 = \langle (inc_1)_h, (inc_1)_a \rangle$ and $inc_2 = \langle (inc_2)_h, (inc_2)_a \rangle$ are such that:

- $(inc_1)_h : \Sigma_C \to \Sigma_D$ and $(inc_2)_h : \Sigma_C \to \Sigma_K$ are the inclusions of Σ_C in Σ_D and Σ_K respectively,
- $(inc_1)_a(\mathbb{A}_D) = \mathbb{A}_D|_{(inc_1)_h}$ and $(inc_2)_a(\mathbb{A}_K) = \mathbb{A}_K|_{(inc_2)_h}$ for $\mathbb{A}_D \in \mathcal{A}_D$ and $\mathbb{A}_K \in \mathcal{A}_K$ (note that both reducts are Σ_C-algebras and so, are in $\mathcal{A}_C$).

The object of the pushout of $\langle inc_1, inc_2 \rangle$ is the fusion of SDL and $\mathcal{K}$ and, up to isomorphism, is such that:

- the signature is Σ_{SDK} that is constituted by Σ_C where C_1 is enriched with the connectives $\Box_D, K_1, \ldots, K_n$ and O_1 is enriched with the operators $\mathbf{N}_D, \mathbf{N}_1, \ldots, \mathbf{N}_n$;
- the set of rules is $\mathcal{R}_{SDK} = \mathcal{R}_C \cup \{\mathrm{L}\mathbf{N}_D\Omega, \mathrm{R}\mathbf{N}_D\Omega, \mathrm{L}\Box_D, \mathrm{R}\Box_D, \mathrm{D}\} \cup \{\mathrm{L}\mathbf{N}_i\Omega, \mathrm{R}\mathbf{N}_i\Omega, \mathrm{L}K_i, \mathrm{R}K_i, \mathrm{T}_i, 4_i, 5_i : i = 1, \ldots, n\}$;
- the class $\mathcal{A}$ of appropriate algebras for $\langle \Sigma_{SDK, \mathcal{R}_{SDK}} \rangle$ such that it includes the class of algebras induced by general Kripke structures (see Definition 3.1.5) equipped with n accessibility relations that are equivalence relations together with a relation that is right-unbounded. ∎

Note that, in the context of the logic system resulting from the fusion described in Example 5.1.3, formulas as $\Box_D K_1 \varphi$ mean that it is obligatory that agent 1 knows φ.

However, as mentioned in the end of Chapter 3, deontic logic should be presented as non-normal modal system. The next example defines a deontic logic system as a non-normal modal logic. In order to differentiate the deontic modality from the previous example with the non-normal deontic modality now presented the latter one will be denoted by $\bigcirc$.

Example 5.1.4. Let $\mathcal{D}_N = \langle \Sigma_N, \mathcal{R}_{DN}, \mathcal{A}_{DN} \rangle$ be a non-normal deontic modal logic system such that:

- Σ_N is as in Definition 3.2.2 where $\Box$ is replaced by $\bigcirc$,
- $\mathcal{R}_{DN}$ is the union of the set $\mathcal{R}_N$ of Definition 3.2.3 and the set $\{\mathrm{C}, \mathrm{N}, \mathrm{D}_{\mathrm{NN}}\}$ of rules presented in Section 3.2,
- $\mathcal{A}_{DN}$ is the class of all Σ_N-algebras appropriate for $\langle \Sigma_N, \mathcal{R}_{DN} \rangle$ which includes the Σ_N-algebras induced by general structures (see Definition 3.2.6) $\langle W, f, A, V \rangle$ such that, for every $w \in W$ and $X, Y \in A$, all the following conditions hold:

  (c) if $w \in f(X)$ and $w \in f(Y)$ then $w \in f(X \cap Y)$

  (d) if $w \in f(X)$ then $w \notin f(W \setminus X)$

  (n) $w \in f(W)$.

Proposition 5.1.5. The Σ_N-algebras induced by general structures described in Example 5.1.4 are in $\mathcal{A}_{DN}$.

Proof. In Theorem 3.2.11 it is established that the induced algebras from general structures for non-normal modal logics are appropriate. It remains to show that each additional rule (C, N and D_{NN}) has its corresponding property (**(c)**, **(n)** and **(d)**) in the general structure.

(rule N) It is needed to show that

$$\text{for every } w \in W, w \in f(W)$$

$$\text{iff}$$

$$\text{for every } \Gamma_1, \Gamma_2, \rho, \alpha \text{ and } \beta, \mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \Gamma_1\rho \to \Gamma_2\rho, \top\rho \sqsubseteq \mathbf{F}(\top\rho).$$

($\Rightarrow$) Assume that, for every $w \in W$, $w \in f(W)$, that is, $f(W) = W$. It is sufficient to show that $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash\to \top\rho \sqsubseteq \mathbf{F}(\top\rho)$ which is the case since $[\![\top\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = W$ and $\mathbf{F}_{\mathbb{A}_{\mathcal{S}}} = \lambda x.f(x)$, using the hypothesis, $[\![\top\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = W \subseteq f(W) = [\![\mathbf{F}(\top\rho)]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta}$.

($\Leftarrow$) Assume that, for every $\Gamma_1, \Gamma_2, \rho, \alpha$ and β, $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \Gamma_1\rho \to \Gamma_2\rho, \top\rho \sqsubseteq \mathbf{F}(\top\rho)$. Then, for each $w \in W$, choose $\Gamma_1, \Gamma_2 = \varnothing$. Therefore $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \top\rho \sqsubseteq \mathbf{F}(\top\rho)$ and so $W \subseteq f(W)$ and in particular $w \in f(W)$.

(rule D_{NN}). It is needed to show that

$$\text{for every } w \in W \text{ and } X \in A \text{ if } w \in f(X) \text{ then } w \notin f(W \setminus X)$$

$$\text{iff}$$

$$\text{for every } \Gamma_1, \Gamma_2, \rho, \alpha \text{ and } \beta, \mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \Omega\tau_1\rho, \tau_1\rho \sqsubseteq \mathbf{F}(\#\xi_1\rho), \Gamma_1 \to \Gamma_2, \tau_1\rho \not\sqsubseteq \mathbf{F}(\#\neg\xi_1\rho).$$

($\Rightarrow$) Assume that, for every $w \in W$ and $X \in A$ if $w \in f(X)$ then $w \notin f(W \setminus X)$. It is sufficient to show that

$$\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \Omega\tau_1\rho, \tau_1\rho \sqsubseteq \mathbf{F}(\#\xi_1\rho) \to \tau_1\rho \not\sqsubseteq \mathbf{F}(\#\neg\xi_1\rho)$$

that is, $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash\to \tau_1\rho \not\sqsubseteq \mathbf{F}(\#\neg\xi_1\rho), \mho\tau_1\rho, \tau_1\rho \not\sqsubseteq \mathbf{F}(\#\xi_1\rho)$. Consider two cases:

a. $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \mho\tau_1\rho$ which immediately establishes the result,

b. otherwise $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \Omega\tau_1\rho$, that is, $[\![\tau_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = \{w\}$ for some $w \in W$. From the definition of $\mathbb{A}_{\mathcal{S}}$, $[\![\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha} = Y$ for some $Y \in A$ and so $[\![\neg\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha} = W \setminus Y$ and since $\#_{\mathbb{A}_{\mathcal{S}}} = \lambda a.a$, $[\![\#\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha} = Y$ and $[\![\#\neg\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha} = W \setminus Y$. From the hypothesis, if $\{w\} \subseteq f(X)$ then $\{w\} \nsubseteq f(W \setminus X)$ for any $X \in A$. So either $[\![\tau_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = \{w\} \nsubseteq f(Y) = \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}([\![\#\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta})$ then $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \tau_1\rho \not\sqsubseteq \mathbf{F}(\#\xi_1\rho)$ or, if $\{w\} \in f(Y)$ then $[\![\tau_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta} = \{w\} \nsubseteq f(W \setminus Y) = \mathbf{F}_{\mathbb{A}_{\mathcal{S}}}[\![\#\neg\xi_1\rho]\!]_{\mathbb{A}_{\mathcal{S}}\alpha\beta}$ so $\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \tau_1\rho \not\sqsubseteq \mathbf{F}(\#\neg\xi_1\rho)$.

($\Leftarrow$) Assume for every $\Gamma_1, \Gamma_2, \rho, \alpha$ and β that

$$\mathbb{A}_{\mathcal{S}}\alpha\beta \Vdash \Omega\tau_1\rho, \tau_1\rho \sqsubseteq \mathbf{F}(\#\xi_1\rho), \Gamma_1 \to \Gamma_2, \tau_1\rho \not\sqsubseteq \mathbf{F}(\# \neg \xi_1\rho).$$

Then, for each $w \in W$ choose $\Gamma_1, \Gamma_2 = \varnothing$ such that $\rho(\tau_1) = \mathbf{x}_1, \rho(\xi_1) = \mathbf{z}_1$ and $\alpha(\mathbf{x}_1) = \{w\}$, $\alpha(\mathbf{z}_1) = X$. Therefore

$$\mathbb{A}_{\mathcal{S}}\alpha \Vdash \tau_1\rho \not\sqsubseteq \mathbf{F}(\#\xi_1\rho)$$

or

$$\mathbb{A}_{\mathcal{S}}\alpha \Vdash \tau_1\rho \not\sqsubseteq \mathbf{F}(\# \neg \xi_1\rho),$$

that is, $\{w\} \nsubseteq f(X)$ or $\{w\} \nsubseteq f(W\backslash X)$. Thus, if $\{w\} \subseteq f(X)$ then $\{w\} \nsubseteq f(W\backslash X)$.

The proof for the rule C is similar. □

In the sequel it is given again a combination between deontic and knowledge logics, but now with the deontic system being a non-normal modal logic. So, in the resulting system, agents can reason about obligations that does not necessarily respect the formula $\bigcirc(\varphi \wedge \psi) \supset (\bigcirc\varphi \wedge \bigcirc\psi)$.

Example 5.1.6. The fusion of $\mathcal{D}_N$ with the system $\mathcal{K}$ of Example 3.1.11 sharing $\mathcal{L}_C$ as in Example 5.1.3 is obtained by taking the morphisms $inc_1 : \mathcal{L}_C \to \mathcal{D}_N$ and $inc_2 : \mathcal{L}_C \to \mathcal{K}$ where $inc_i = \langle (inc_i)_h, (inc_i)_a \rangle$ for $i = 1, 2$ are such that

- $(inc_1)_h : \Sigma_C \to \Sigma_N$ and $(inc_2)_h : \Sigma_C \to \Sigma_K$ are the obvious inclusions,
- $(inc_1)_a(\mathbb{A}_{DN}) = \mathbb{A}_{DN}|_{(inc_1)_h}$ and $(inc_2)_a(\mathbb{A}_K) = \mathbb{A}_K|_{(inc_1)_h}$ for $\mathbb{A}_{DN} \in \mathcal{A}_{DN}$ and $\mathbb{A}_K \in \mathcal{A}_K$.

The object of the pushout of $\langle inc_1, inc_2 \rangle$ is the fusion of $\mathcal{D}_N$ and $\mathcal{K}$ and, up to isomorphism, is such that (the modality from $\mathcal{D}_N$ is herein renamed to $\bigcirc$ in order to differentiate this system from the one resulting of the fusion of the previous example):

- the signature is Σ_{NDK} that is constituted by Σ_C where C_1 is enriched with the modalities $\bigcirc, K_1, \ldots, K_n$ and O_1 is enriched with the operators $\mathbf{F}, \mathbf{N}_1, \ldots, \mathbf{N}_n$;
- the set of rules is $\mathcal{R}_{NDK} = \mathcal{R}_C \cup \{\text{L}\bigcirc, \text{R}\bigcirc, \text{eq}\mathbf{F}, \text{C}, \text{N}, \text{D}_{\text{NN}}\} \cup \{\text{L}\mathbf{N}_i\Omega, \text{R}\mathbf{N}_i\Omega, \text{L}K_i, \text{R}K_i, \text{T}_i, 4_i, 5_i : i = 1, \ldots, n\}$;
- the class of algebras of the logic resulting from the combination is the class of appropriate algebras for $\langle \Sigma_{NDK}, \mathcal{R}_{NDK} \rangle$ which includes the algebras induced by general "mixed" structures of the form $\langle W, f, R_1, \ldots, R_n, \mathcal{B}, V \rangle$ where each R_i is an equivalence relation for $i = 1, \ldots, n$ and f respects conditions **(c)**, **(d)** and **(n)**. ■

The possibility of sharing modalities is now illustrated with an example of the fusion of two well known normal modal logics: a logic where the accessibility relation of the modality is reflexive and other where the accessibility relation is transitive.

Example 5.1.7. Let $\mathcal{L}_4 = \langle \Sigma_M, \mathcal{R}_M \cup \{4\}, \mathcal{A}_4 \rangle$ be the modal logic system where 4 is the rule 4_i in Table 3.3 and $\mathcal{A}_4$ is the class of all algebras (modulo equivalence) induced by general Kripke structures (see Definition 3.1.5) over transitive accessibility relations, and $\mathcal{L}_T = \langle \Sigma_M, \mathcal{R}_M \cup \{\mathrm{T}\}, \mathcal{A}_T \rangle$ be the modal logic system where T is the rule T_i in Table 3.3 and $\mathcal{A}_T$ is the class of all algebras (modulo equivalence) induced by general Kripke structures (see Definition 3.1.5) over reflexive accessibility relations.

Take $\mathcal{L}_C = \langle \Sigma_M, \mathcal{R}_M, \mathcal{A}_M \rangle$, the "common logic system" in the fusion according to Definition 4.2.11, that is Σ_M is the signature described in Definition 3.1.1, $\mathcal{R}_M$ is the set of rules described in Definition 3.1.2, and $\mathcal{A}_M$ is the class of Σ_M-algebras full for $\langle \Sigma_M, \mathcal{R}_M \rangle$. That is, $\mathcal{L}_C$ is the normal modal logic system known in the literature as K.

The fusion of $\mathcal{L}_4$ and $\mathcal{L}_T$ sharing $\mathcal{L}_C$ is obtained by taking the morphisms $inc_1 : \mathcal{L}_C \to \mathcal{L}_4$ and $inc_2 : \mathcal{L}_C \to \mathcal{L}_T$ where $inc_i = \langle (inc_i)_h, (inc_i)_a \rangle$ for $i = 1, 2$ are such that:

- $(inc_i)_h : \Sigma_M \to \Sigma_M$ for $i = 1, 2$ is the obvious bijection;
- $(inc_1)_a(\mathbb{A}_4) = \mathbb{A}_4|_{(inc_1)_h}$ for $\mathbb{A}_4 \in \mathcal{A}_4$ and $(inc_2)_a(\mathbb{A}_T) = \mathbb{A}_T|_{(inc_2)_h}$ for $\mathbb{A}_T \in \mathcal{A}_T$.

So, the resulting system is $\mathcal{L}_{S4} = \langle \Sigma_M, \mathcal{R}_M \cup \{4, \mathrm{T}\}, \mathcal{A}_{S4} \rangle$ where $\mathcal{A}_{S4}$ is the class of appropriate algebras for the calculus of $\mathcal{L}_{S4}$ which includes the algebras induced by general Kripke structures such that the accessibility relation is transitive and reflexive, commonly known as the S4 modal system. ∎

Example 5.1.7 shows the importance of $\mathcal{L}_C$. If $\mathcal{L}_C$ was a system without the modality, as the one in Example 5.1.3, then the resulting combination was the bimodal logic system where one modality has a reflexive relation and the other has a transitive relation.

It is also possible to share non-normal modalities, as the next example illustrates.

Example 5.1.8. Let $\mathcal{M} = \langle \Sigma_{MT}, \mathcal{R}_{MT}, \mathcal{A}_{MT} \rangle$ be the monotonic non-normal modal logic of Example 3.2.20 and $\mathcal{C}_{NN} = \langle \Sigma_N, \mathcal{R}_{CN}, \mathcal{A}_{CN} \rangle$ a non-normal modal logic system where Σ_N is the non-normal modal signature of Definition 3.2.2, $\mathcal{R}_{CN} = \mathcal{R}_{NN} \cup \{\mathrm{C}\}$ with $\mathcal{R}_{NN}$ being the minimum set of rules described in Definition 3.2.3

and C the rule

$$\frac{}{\Omega\tau_1, \tau_1 \sqsubseteq \mathbf{F}(\#\xi_1), \tau_1 \sqsubseteq \mathbf{F}(\#\xi_2), \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{F}(\#\xi_1 \wedge \xi_2)},$$

and $\mathcal{A}_{CN}$ is the class of Σ_N-algebras full for $\langle \Sigma_N, \mathcal{R}_{CN} \rangle$.

The fusion of $\mathcal{M}$ and $\mathcal{C}_{NN}$ sharing E (the smallest non-normal system that can be defined in Definition 3.2.19) where inc_1 and inc_2 are the obvious inclusions is the system $\mathcal{R}eg$ of regular systems defined in Example 3.2.21. ∎

5.2 Preservation of Soundness and Completeness

In this section preservation of soundness and completeness by fusion is analysed for some of the examples of the previous section. More specifically it is analysed the fusion of the SDL and $\mathcal{GL}$ logic systems described in Example 5.1.1, the fusion of the $\mathcal{K}$ and SDL logic systems presented in Example 5.1.3, and the fusion of the normal and non-normal systems in Example 5.1.6. After that, the logic system described in [FF05] (counter-example of the strategy of define fusion of non-normal modal logics as successive modalizations) is presented as LSLTV and it is shown that the logic system resulting from its fusion with itself is sound and complete.

Example 5.2.1. Consider the logic system resulting from the fusion of the standard deontic logic system and the provability logic system, presented in Example 5.1.1. Both component logic systems are sound by Corollary 2.5.2, so, by Theorem 4.3.1 their fusion is also sound. The logic system resulting from the fusion is complete, by Theorem 4.3.2, since all the rules of the standard deontic and Löb provability systems are endowed with persistent provisos and both classes of algebras are full.

Now consider the logic system resulting from the fusion of knowledge logic and a standard deontic logic as described in Example 5.1.3. The knowledge and the deontic systems are sound (again by Corolary 2.5.2), so the logic system resulting from the fusion is also sound. Moreover it is complete by Theorem 4.3.2 since all the rules of knowledge and deontic systems are endowed with persistent provisos and their classes of algebras are full.

Finally, with respect to Example 5.1.6 that presents the fusion of the non-normal deontic system $\mathcal{D}_N$ with the knowledge logic system $\mathcal{K}$ the results are analogous: both systems involved in the fusion are sound, all rules are endowed with persistent provisos and all classes of algebras are full, so the logic system resulting from the fusion is sound and complete. ∎

The non-normal modal partition logic $\mathcal{M}_P$ was defined in [FF05] as a non-normal modal system containing the axiom

$$\text{P} \quad (\Box\varphi \Leftrightarrow \varphi) \vee (\Box\psi \Leftrightarrow \neg\,\psi).$$

This system is particularly interesting for fusion since its combination with itself contains strong interactions between the modalities as, for instance, commutativity ($\Box_1\Box_2\varphi \Leftrightarrow \Box_2\Box_1\varphi$). It is now shown that the preservation of soundness and completeness by the fusion of $\mathcal{M}_P$ with itself is obtained directly by the results established in this thesis.

Example 5.2.2. Let $\mathcal{M}_{P_i} = \langle \Sigma_{MP_i}, \mathcal{R}_{MP_i}, \mathcal{A}_{MP_i} \rangle$ with $i = 1, 2$ be the non-normal modal logic systems where:

- Σ_{MP_i} is the signature Σ_N of Definition 3.2.2,
- $\mathcal{R}_{MP_i}$ is the union of the set $\mathcal{R}_N$ of Definition 3.2.3 with the set of rules presented in Table 5.1,
- $\mathcal{A}_{MP_i}$ is the class of all Σ_{MP_i}-algebras appropriate for $\langle \Sigma_{MP_i}, \mathcal{R}_{MP_i} \rangle$ which includes the algebras induced by general structures $\langle W, f, A, V \rangle$ (see Definition 3.2.6) such that, for every $w \in W$,

  either $w \in f(X)$ for all $X \subseteq W$ such that $w \notin X$,

  or $w \in f(X)$ for all $X \subseteq W$ such that $w \in X$.

$$\text{RP} \quad \frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \tau_2 \quad \Omega\tau_1, \tau_1 \not\sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \mathbf{F}(\tau_3)}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{F}(\tau_2)}$$

$$\text{LP} \quad \frac{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \tau_2 \quad \Omega\tau_1, \tau_1 \sqsubseteq \tau_3, \Gamma_1 \to \Gamma_2, \tau_1 \sqsubseteq \mathbf{F}(\tau_3)}{\Omega\tau_1, \Gamma_1 \to \Gamma_2, \tau_1 \not\sqsubseteq \mathbf{F}(\tau_2)}$$

Table 5.1: Rules for the non-normal partition logic

Take $\mathcal{L}_C$ as in Example 5.1.3 and $inc_i = \langle (inc_i)_h, (inc_i)_a \rangle$ be such that $(inc_i)_h : \Sigma_C \to \Sigma_{MPi}$ is the obvious inclusion and $(inc_i)_a(\mathbb{A}_i) = \mathbb{A}_i|_{(inc_i)_h}$ for $\mathbb{A}_i \in \mathcal{A}_{MP_i}$.

The resulting system of the fusion by sharing $\mathcal{L}_C$ of $\mathcal{M}_{P_1}$ and $\mathcal{M}_{P_2}$ is sound by Theorem 4.3.1 since each $\mathcal{M}_{P_i}$ is sound by Corollary 2.5.2 and is complete by Theorem 4.3.2 since all rules are endowed with persistent provisos and both classes of algebras are full. ■

Chapter 6

Conclusion

As the epilogue of this thesis, some remarks are made in Section 6.1, discussing some topics that can open new directions to the research outlined in Section 6.2.

6.1 Final Remarks

In Chapter 2 the framework used during the thesis was presented. LSLTV was already used in [MSSV04] to present normal modal logic systems but herein a subtle modification was made in the presentation incorporating the set $\mathbb{P}$ in the signature of the language. This was made to allow the framework to be able to capture in a more efficient way different structures of truth values that, for instance, do not have the notion of atomic values as can be the case of some many-valued logics. The interesting point about this change in the language is the possibility to capture the exact language of labelled systems where labels are simple worlds [Vig00]. In this case the signature for modal logic would be the tuple $\langle C, O, \mathbb{P}, X, Y, Z \rangle$ where C, X, Y and Z are as in Definition 3.1.1 but O would have only $\top, \perp \in O_0$ and $O_k = \varnothing$ for all $k \geq 1$ and $\mathbb{P}$ would have only a binary relation R that models directly the accessibility relation in a Kripke structure.

Chapter 3 presented the general class of modal logic systems: both normal and non-normal are defined in a modular calculus which constitutes a novelty for the non-normal case, achieving one of the aims of this work. Using the same principles it is possible to describe several logic systems using LSLTV. Intuitionistic and relevance logics are some cases that can be studied in this context after doing perhaps some minor adaptations. For intuitionistic logic, the signature could be defined as the signature for normal modal logic systems without the connective $\square$ (but maintaining the operator **N**). In terms of rules, rules for reflexivity and transitivity (T and 4) of the relation must be in the calculus as well as a rule capturing the persistence property. The specific rules for $\supset$ and $\neg$ connectives also need to be modified. For relevance logics, that also have a "Kripke-like" semantics, the fact that they are substructural logics require special attention: most of the results presented here should be adapted, as, for instance, the completeness result.

Chapter 4 is the core of the thesis, where the aims of defining fusion of LSLTV and the study about the preservation of soundness and completeness are attained. Part of these results were already been published in [RRS10]. The use of a categorical approach for fusion led to the possibility of using sets of shared symbols different from the usual ones in the literature (namely the propositional connectives). However the definition presented in this work is still restricted in some cases due to the morphisms that are used. This is expected since fusion is in fact a restricted kind of combination. For instance, there is no morphism between a normal modal system and a non-normal modal system (the operators **N** and **F** cannot be mapped one to the other) for the definition used in this work.

Chapter 5 attains the final goal of the thesis. Fusion of two modal logics where at least one of them is non-normal is defined.

6.2 Future Work

Although the categorical definition of fusion in Definition 4.2.11 can be applied to logic systems that are not necessarily (normal) modal systems (generalizing the scope of application of the algebraic definition of fusion), it is possible to generalize even more the class of logic systems to which fusion can be applied. Some of the generalizations that could be proposed includes:

- generalization of the universe of logic systems involved in the combination;
- generalization of the common logic system $\mathcal{L}_C$;
- generalization of the conditions for a pair of logic systems to be eligible for combination.

An interesting direction would be to investigate whether or not the generalizations preserve the results established in Section 4.3.

To what concerns the universe of logic systems involved in the combination, the idea is to generalize Definition 4.2.11 by allowing structural logic systems. Observe that this generalization brings the combination to the border of what can be considered fusion since it is applied to systems not necessarily modal.

The second generalization consists in removing the condition $\Sigma_P \subseteq \Sigma_C$. That is, to allow Σ_C to be a smaller set than Σ_P while assuming that the morphism to each component logic is a monomorphism, so an injection of Σ_C to each Σ_i. More rigorously, Σ_C should be non-empty and should exists an injection from Σ_C to each Σ_i with $i = 1, 2$.

The third generalization is about the set of rules $\mathcal{R}_C$ and consists on removing the condition that it is a subset of each $\mathcal{R}_i$ while imposing that there are rules in $\mathcal{R}_C$ for the shared connectives and operators. This generalization changes the object of the pushout given before.

In what respects the categorical approach, different definitions of the morphism between LSLTV can be investigated. One idea is to modify the mapping between algebras, allowing different sets of sorts between the source and target algebra.

Another direction of research is to investigate preservation of properties besides soundness and completeness. It is possible that the techniques for proving preservation of Craig interpolation presented in [CRS08b] can be adapted to the context of labelled sequents.

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