Analysis and Synthesis of Logics

How to Cut and Paste Reasoning Systems

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Preface

The aim of this book is to show how logics can be cut and paste in order to be applied to express and model problems in several distinct areas. The universal applicability of logic in both pure and applied science is a fact that defies philosophers. Contemporary logical research, however, has an undeniable tendency towards pluralism and compartimentation, as shown by the division of philosophical logic in areas and subfields. On the one hand, we have logics alternative to classic, such as many-valued logic, intuitionistic logic, paraconsistent logic. On the other hand, we also have logics complementary to classic, such as modal logics, and, in particular, temporal logic, epistemic logic, doxastic logic, erotetic logic, deontic logic, and so on.

Considering that reasoning is through process, this compartimentation, even if driven by methodological and technical reasons, has been said to be harmful to logic while a philosophical discipline. From this viewpoint, combinations of logics goes in the opposite direction of restoring the entirety of logic as wide theory of rationality, much in the same spirit to what happens in areas as algebraic geometry. Thus, from a philosophical perspective, logical combinations of tense and modality, for instance, may offer a better look to issues in the theory of causation and action. Combining temporal logic with alethic modal logics adds a temporal dimension to knowledge and belief.

Conceptually, the idea of looking to logic as an entirety instead of isolated fragments is not new. Philosophers and logicians from Ramón Lull (1235-1316), in his *Ars Magna*, to Gottfried W. Leibniz (1646–1716), with his *calculus ratiocinator* [175], have dreamed of building schemes or even machines that can reason by combining different logics or logic-like mechanisms that could cooperate instead of competing.

The activity of combining logics, as seen nowadays, offers an important tool for modularity. Rather than building a logic from scratch, it may be better for some applications to depend upon previous work on specialized topics. The underlying idea is that logics can be reusable, leading to a perspective gain with the resulting combined system. However, there are many technical difficulties if one is interested in the practical activity of combining logics. Symbols may mean different things in different logics. How is it possible to define the languages in order to compose them into an organic entity? Also, proofs and derivations can have different meanings in different logics. How to thread rules and derivation schemes of totally different nature?

This book intends to address these questions in detail, presenting with the foremost rigor the issues of logical manipulation. The reader will learn here how to set up the syntactical dimension in detail, and how to define the semantics and the proof theory for recombinant logics. The impact of combination of logics in practice can only be assessed by people involved in the application domains. However, we believe that these techniques can be useful in fields such as computational linguistics, automated theorem proving, complexity and artificial intelligence. Other promising applications are in the areas of software specification, knowledge representation, architectures for intelligent computing and quantum computing, security protocols and authentication, secure computation and zero-knowledge proof systems and in the formal ethics of cryptographic protocols.

Combinations of temporal reasoning, reasoning in description logic, reasoning about space and distance are becoming a relevant toolbox in modeling multi-agent systems. The resulting hybrid systems have the main advantage of combining logics which would be otherwise incompatible. Proof procedures with controlled complexity, model checking and satisfiability checking procedures can be obtained for a bigger logic from the respective procedures for the component logics.

But the reader should not think that combinations of logics is a topic restricted to applications outside logic. On the contrary, although we do not deal with this question in this book, the very idea of combining logics, as we see it, touches on more abstract domains as applied to the logical theory itself: for instance, as suggested in [230]. The idea of combining logics can be even useful to understand apparently far away topics such as Popper's structuralist theory of logic, as in [220], where an elementary theory of combining negations was developed.

In a rigorous way, the problem of combining logics can be seen as follows: given two logics $\mathcal{L}_1$ and $\mathcal{L}_2$ we want to combine them and obtain a new logic $\mathcal{L}$ satisfying certain requirements. In general, there are several mechanisms to combine the original logics. Choosing mechanism $\odot$, the new logic is $\mathcal{L} = \mathcal{L}_1 \odot \mathcal{L}_2$. That is, $\odot$ is an operator on some class of logics including $\mathcal{L}_1$ and $\mathcal{L}_2$. Different operators may lead to different resulting logics. Most of the operators provide an algorithmic construction of logic $\mathcal{L}$ by stating its language, semantic structures and/or deductive systems. Moreover, the construction of $\mathcal{L}$ usually is a minimal (or maximal) construction. The combined logic should extend the components in a controlled way, so that it does not include undesirable features.

All the mechanisms assume that the component logics are presented in the same way. In technical terms we say they are homogeneous. For instance, both of them are presented by Hilbert calculi. However, some assume that component logics need some preparation before being combined. For instance, assume that we say that component logics are endowed with an algebraic semantics. In this case, we have to say how the semantic structures of the component logics induce an algebraic semantics. In the book we deal with heterogeneous fibring in a moderated way in Chapter 3 and with heterogeneous fibring of deductive systems in Chapter 4.

One of the most challenging problems is related to proving transference results. That is, to investigate sufficient conditions for the preservation of properties, namely soundness, completeness, decidability, consistency, interpolation, from the components into the resulting logic.

Combination mechanisms can be extended to a finite number of components and sometimes even to an infinite number of components.

Among the different combination mechanisms we can refer to fibring which is one of the objects of this book. Fusion, if not historically the first, is the simplest method, and the best studied combination mechanism.

Combining logics in the perspective of this book does not mean only synthesizing or composing logics (which is called splicing), but is also intended to work in the opposite direction of decomposing logics, called splitting. Herein, we analyze the possible-translations semantics mechanism.

The idea of writing this book originated during *The Workshop on Combination of Logics: Theory and Applications* (CombLog04) [48, 50], held in the Center for Logic and Computation, at the Department of Mathematics of IST, Technical University of Lisbon, Portugal, from July 28-30, 2004. Encouraged by the vigor of the field and by the interest triggered by this and several other conferences (such as [231, 109, 159, 7, 135]), we decided to accept the challenge to produce a book containing some basic ideas, methods and techniques that could help logicians, computer scientists and philosophers to have access to a general yet elementary informal theory of combinations of logics. The book intended to bring together a sample of results, problems and perspectives involving the idea of cutting and pasting logics, explaining when possible the role of the underlying constructions as universal arguments in the categorial sense.

We depart here from a basic universe of logic systems starting with propositional-based systems endowed with Hilbert calculi and ordered algebraic semantics. This basic setting is already rich enough to encompass interesting features of fibring with several applications and to provide the basic techniques for the trade of combining systems. Later on we extend the notion to the first-order and to the higher-order domains.

Chapter 1 is an introductory overview to the essential ingredients of composing and decomposing logics. In Section 1.1, we introduce the concept of consequence system as the basic abstraction to describe a logic system. In Section 1.2, we present the basic ideas about composing or splicing logics and decomposing or splitting logics. We also introduce a technical summary of some combination mechanisms like fusion, product and fibring by functions of modal logics. We also refer to Gödel provability logic as an illustration of a splitting mechanism. In Section 1.3, we provide a very brief introduction to algebraic fibring using Hilbert calculi. In Section 1.4, we sketch the splitting mechanism called possible-translations semantics.

Chapter 2 concentrates on fibring of propositional based logics presented as Hilbert calculi. Moreover, some preservation results are introduced. In Section 2.1, signatures, and their fibring are presented. In Section 2.2, we dedicate our attention to the fibring of Hilbert calculi. We illustrate the concepts with several examples including classical logic, modal logics, intuitionistic logic, 3-valued Gödel and Łukasiewicz logics. In Section 2.3, we discuss several preservation results. Finally, Section 2.4 presents some final remarks.

Chapter 3 is dedicated to the fibring of semantics for propositional based logics. Ordered algebras are the basic semantic structures adopted. We also include the relationship to fusion and fibring by functions. Again some preservation results are given. In Section 3.1, we introduce the semantic structures and their fibring. We illustrate the concepts with several examples including classical logic, modal logics, intuitionistic logic, 3-valued Gödel and Łukasiewicz logics. In Section 3.2, we present the notions of logic system, soundness and completeness. In Section 3.3, we discuss the preservation of soundness and completeness properties. In Section 3.4, we establish the relationship between the present approach and fibring by functions. In Section 3.5 we present some final comments.

Chapter 4 is dedicated to the analysis of fibring of logics that are not presented in the same way. Two solutions are proposed. The first one is based on fibring of consequence systems and the second one on abstract proof systems. Some preservation results are established. Section 4.1 concentrates on fibring of consequence systems using a fixed point operator. Several examples are given for logics presented either in a proof-theoretic or a model-theoretic way. Section 4.2 focuses on the notion of abstract proof system and looks at the proof systems induced by Hilbert, sequent and tableau calculi. Moreover, it includes the notion of fibring of abstract proof systems. We also discuss some relationships between consequence systems and proof systems. In Section 4.3 we present some final remarks.

Chapter 5 studies composition of non-truth functional logics via fibring, an important extension of the theory, considering that many of the interesting logics for applications are not truth-functional. In Section 5.1, the notion of interpretation system presentation is introduced. In Section 5.2 the notions of unconstrained and constrained fibring of interpretation system presentations is defined. In Section 5.3 we again use Hilbert calculus as the suitable proof-theoretic notion. In Section 5.4 some preservation results are established, namely, the preservation of soundness and completeness. Section 5.5 discusses self-fibring in the context of non-truth-functional logics. In Section 5.6 we present some final comments.

Chapter 6 concentrates on fibring of first-order based logics. It can be seen as an extension of the fibring of propositional based logics, choosing particular powerset algebras as semantic structures. The running example is fibring of classical first-order logic and modal logic. In Section 6.1, first-order based signatures and the corresponding languages are introduced. Next, in Section 6.2, we present interpretation structures and interpretation systems. First-order Hilbert calculi are presented in Section 6.3. Section 6.4 introduces first-order logic systems. Then, in Section 6.5, we define fibring of first-order based logics. The preservation of completeness and other metatheorems by fibring is discussed in Section 6.6, where we also briefly sketch a proof of completeness for a particular class of first-order logic systems. In Section 6.7 we make some final remarks.

Chapter 7 deals with higher-order quantification logics. The semantic structures are generalizations of the usual topos semantics for higher-order logics. In Section 7.1 we introduce the relevant signatures. In Section 7.2 the Hilbert calculus. In Section 7.3 is dedicated to setting up the semantic notions. Section 7.4 introduces the notion of logic system, and we briefly discuss some related notions such as soundness and completeness. The novelty here is that the usual notion of soundness must be modified in the present framework. In Section 7.5, a general completeness theorem is established. In Section 7.6, the notions of constrained and unconstrained fibring of logic systems are given, and it is shown that soundness is preserved by fibring and a completeness preservation result is obtained. In Section 7.7 we briefly discuss the main results described in the chapter.

In Chapter 8, we turn our attention to modulated fibring. This variant was developed to cope with collapsing problems: in some cases when two logics are combined one of them collapses with the other. We illustrate the concepts with examples including propositional logic, intuitionistic logic, 3-valued Gödel and Łukasiewicz logics. In Section 8.1, we introduce the notions of modulated signature and modulated signature morphisms. In Section 8.2, we describe modulated interpretation structures, modulated interpretation systems and the corresponding morphisms. Next we present the notion of bridge between modulated interpretation systems. In Section 8.3, we define modulated Hilbert calculus and their morphisms. In Section 8.4 is dedicated to modulated logic systems and their corresponding morphisms. In Section 8.5, we establish soundness and completeness preservation results. Finally, Section 8.6 presents some final comments.

Chapter 9 introduces the problem of splitting logics, emphasizing the role of possible-translations semantics and contrasting with the previous chapters that deal with forms of splicing. In Section 9.1, a category of propositional based signatures suitable for splitting logics is introduced, as well as the corresponding category of consequence systems. In Section 9.2 the technique known as possible-translations characterization is analyzed, and some applications are given. In Section 9.3 two methods for combining matrix logics, plain fibring and direct union of matrices, are reviewed. Finally, Section 9.4 presents some final comments.

In Chapter 10 we discuss new tendencies on fibring. In Section 10.1, we motivate network fibring using modal logic. In Sections 10.2, 10.3, 10.4 and 10.5, some case-studies are introduced. Section 10.2 discusses integration of information flows by describing a system in which reasoning and proofs from different sources of information can be accommodated. In Section 10.3, we refer to some generalizations of logic input/output operations. We also discuss how to combine input/output operations into networks. In Section 10.4, we discuss the fibring of neural networks. In Section 10.5, we turn our attention to recursive Bayesian networks. In Section 10.6, the notion of self-fibring of networks is introduced. Section 10.7 presents some concluding remarks.

Finally, in Chapter 11 we first present a summing-up of the different techniques for combining logics presented in this book together with their main features. Then we move to a brief overview of applications where fibring can be directly used, as well as to emergent fields of application. It also includes an outlook of new research directions in both the existing combination mechanisms but also to new forms of combination.

We observe that most chapters of the book deal with combination of logics rather than with decomposition of logics. This happens because splitting mechanisms are not so well developed.

We assume that the readers are familiar with basic logic notions of classical propositional and first-order logics at the level of, for instance, [203] and [88] and propositional modal logics at the level of, for instance, [150]. Although not mandatory, a very basic knowledge of categories (see [183]) is useful for better understanding the minimality of the constructions.

The book is intended to be a research monograph for those that want to know the state-of-the art in composing and decomposing logics, that want to know about issues worthwhile to be pursued, as well as potential contemporary applications of these techniques. If you are one of these we recommend that you have the patience to read the whole book. If you want to focus on particular aspects of combination of logics, we suggest several paths hoping that one of them is of your taste.

- If the reader is only interested in knowing what are the main issues in the combination of logics, we recommend Chapters 1, 2 and 3 which provide a basic account on consequence systems and the basic notions of propositional fibring;
- If you are curious about decomposition and its importance to non-truth functional logics you should read Chapters 1 and 9 and maybe it is useful reading Chapter 5;
- The reader interested in a more general form of fibring (capturing more propositional-based logics) and avoiding the well known collapsing problem should concentrate on Chapter 8, besides Chapters 1, 2 and 3;
- Someone with research interest in proof systems and how to combine different proof systems should read Chapters 1, 2 and 4;
- If your interests are in modal logic, we suggest you read Chapters 1, 2, 3, 4 and 6;
- If you are a first-order logician and have curiosity on combination of logics, we suggest you read Chapters 1, 2, 3 and, more importantly, Chapter 6;
- If you are a higher-order logician and want to grasp what is combination of logics, we suggest you read Chapters 1, 2, 3 and, of course, Chapter 7;
- If you are an intuitionistic logician and would like to know about combination of logics, we suggest you read Chapters 1, 2, 3 and Chapter 8;
- If you like to know the potential of combination in contexts that are not logical in nature you should read Chapter 10.

For a summing-up of the techniques used in the book, as well as some applications and topics for further research, we recommend Chapter 11.

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